1 INTRODUCTION

1.1 Delimiting the topic

The title of this piece is not ‘A Survey of Relevance Logic’. Such a project was impossible in the mid 1980s when the first version of this article was published, due to the development of the field and even the space limitations of the Handbook. The situation is if anything, more difficult now. For example Anderson and Belnap and Dunn’s two volume [1975, 1992] work Entailment: The Logic of Relevance and Necessity, runs to over 1200 pages, and is their summary of just some of the work done by them and their co-workers up to about the late 1980s. Further, the comprehensive bibliography (prepared by R. G. Wolf) contains over 3000 entries in work on relevance logic and related fields.

So, we need some way of delimiting our topic. To be honest the fact that we are writing this is already a kind of delimitation. It is natural that you shall find emphasised here the work that we happen to know best. But still rationality demands a less subjective rationale, and so we will proceed as follows.

Anderson [1963] set forth some open problems for his and Belnap’s system E that have given shape to much of the subsequent research in relevance logic (even much of the earlier work can be seen as related to these open problems, e.g. by giving rise to them). Anderson picks three of these problems as major: (1) the admissibility of Ackermann’s rule $\gamma$ (the reader should not worry that he is expected to already know what this means), (2) the decision problems, (3) the providing of a semantics. Anderson also lists additional problems which he calls ‘minor’ because they have no ‘philosophical bite’. We will organise our remarks on relevance logic around three major problems of Anderson. The reader should be told in advance that each of these problems are closed (but of course ‘closed’ does not mean ‘finished’—closing one problem invariably opens another related problem). This gives then three of our sections. It is obvious that to these we must add an introduction setting forth at least some of the motivations of relevance logic and some syntactical specifications. To the end we will add a section which situates work in relevance logic in the wider context of study of other
logical systems, since in the recent years it has become clear that relevance logics fit well among a wider class of ‘resource-conscious’ or ‘substructural’ logics [Schroeder-Heister and Dosen, 1993, Restall, 2000] [and cite the S–H article in this volume]. We thus have the following table of contents:

1. Introduction
2. The Admissibility of γ
3. Semantics
4. The Decision Problem
5. Looking About

We should add a word about the delimitation of our topic. There are by now a host of formal systems that can be said with some justification to be ‘relevance logics’. Some of these antedate the Anderson–Belnap approach, some are more recent. Some have been studied somewhat extensively, whereas others have been discussed for only a few pages in some journal. It would be impossible to describe all of these, let alone to assess in each and every case how they compare with the Anderson–Belnap approach. It is clear that the Anderson–Belnap-style logics have been the most intensively studied. So we will concentrate on the research program of Anderson, Belnap and their co-workers, and shall mention other approaches only insular as they bear on this program. By way of minor recompense we mention that Anderson and Belnap [1975] have been good about discussing related approaches, especially the older ones.

Finally, we should say that our paradigm of a relevance logic throughout this essay will be the Anderson–Belnap system R or relevant implication (first devised by Belnap—see [Belnap, 1967a, Belnap, 1967b] for its history) and not so much the Anderson–Belnap favourite, their system E of entailment. There will be more about each of these systems below (they are explicitly formulated in Section 1.3), but let us simply say here that each of these is concerned to formalise a species of implication (or the conditional—see Section 1.2) in which the antecedent suffices relevantly for the consequent. The system E differs from the system R primarily by adding necessity to this relationship, and in this E is a modal logic as well as a relevance logic. This by itself gives good reason to consider R and not E as the paradigm of a relevance logic.¹

¹It should be entered in the record that there are some workers in relevance logic who consider both R and E too strong for at least some purposes (see [Routley, 1977], [Routley et al., 1982], and more recently, [Brady, 1996]).
1.2 Implication and the Conditional

Before turning to matters of logical substance, let us first introduce a framework for grammar and nomenclature that is helpful in understanding the ways that writers on relevance logic often express themselves. We draw heavily on the ‘Grammatical Propaedeutic’ appendix of [Anderson and Belnap, 1975] and to a lesser extent on [Meyer, 1966], both of which are very much recommended to the reader for their wise heresy from logical tradition.

Thus logical tradition (think of [Quine, 1953]) makes much of the grammatical distinction between ‘if, then’ (a connective), and ‘implies’ or its rough synonym ‘entails’ (transitive verbs). This tradition opposes

1. If today is Tuesday, then this is Belgium

to the pair of sentences

2. ‘Today is Tuesday’ implies ‘This is Belgium’,

3. That today is Tuesday implies that this is Belgium.

And the tradition insists that (1) be called a conditional, and that (2) and (3) be called implications.

Sometimes much philosophical weight is made to rest on this distinction. It is said that since ‘implies’ is a verb demanding nouns to flank it, that implication must then be a relation between the objects stood for by those nouns, whereas it is said that ‘if, then’ is instead a connective combining that implication (unlike ‘if, then’) is really a metalinguistic notion, either overtly as in (2) where the nouns are names of sentences, or else covertly as in (3) where the nouns are naming propositions (the ‘ghosts’ of linguistic entities). This last is then felt to be especially bad because it involves ontological commitment to propositions or some equally disreputable entities. The first is at least free of such questionable ontological commitments, but does raise real complications about ‘nested implications’, which would seem to take us into a meta-metalanguage, etc.

The response of relevance logicians to this distinction has been largely one of ‘What, me worry?’ Sometime sympathetic outsiders have tried to apologise for what might be quickly labelled a ‘use–mention confusion’ on the part of relevance logicians [Scott, 1971]. But ‘hard-core’ relevance logicians often seem to luxuriate in this ‘confusion’. As Anderson and Belnap [1975, p. 473] say of their ‘Grammatical Propaedeutic’: “the principle aim of this piece is to convince the reader that it is philosophically respectable to ‘confuse’ implication or entailment with the conditional, and indeed philosophically suspect to harp on the dangers of such a ‘confusion’. (The suspicion is that such harpists are plucking a metaphysical tune on merely grammatical strings.)”
The gist of the Anderson–Belnap position is that there is a generic conditional-implication notion, which can be carried into English by a variety of grammatical constructions. Implication itself can be viewed as a connective requiring prenominalisation: ‘that ___ implies that ___’, and as such it nests. It is an incidental feature of English that it favours sentences with main subjects and verbs, and ‘implies’ conforms to this reference by the trick of disguising sentences as nouns by prenominalisation. But such grammatical prejudices need not be taken as enshrining ontological presuppositions.

Let us use the label ‘Correspondence Thesis’ for the claim that Anderson and Belnap come close to making (but do not actually make), namely, that in general there is nothing other than a purely grammatical distinction between sentences of the forms

4. If $A$, then $B$, and

5. That $A$ implies that $B$.

Now undoubtedly the Correspondence Thesis overstates matters. Thus, to bring in just one consideration, [Castañeda, 1975, pp. 66 ff.] distinguishes ‘if $A$ then $B$’ from ‘$A$ only if $B$’ by virtue of an essentially pragmatic distinction (frozen into grammar) of ‘thematic’ emphases, which cuts across the logical distinction of antecedent and consequent. Putting things quickly, ‘if’ introduces a sufficient condition for something happening, something being done, etc. whereas ‘only if’ introduces a necessary condition. Thus ‘if’ (by itself or prefixed with ‘only’) always introduces the state of affairs thought of as a condition for something else, then something else being thus the focus of attention. Since ‘that $A$ implies that $B$’ is devoid of such thematic indicators, it is not equivalent at every level of analysis to either ‘if $A$ then $B$’ or ‘$A$ only if $B$’.

It is worth remarking that since the formal logician’s $A \rightarrow B$ is equally devoid of thematic indicators, ‘that $A$ implies that $B$’ would seem to make a better reading of it than either ‘if $A$ then $B$’ or ‘$A$ only if $B$’. And yet it is almost universally rejected by writers of elementary logic texts as even an acceptable reading.

And, of course, another consideration against the Correspondence Thesis is produced by notorious examples like Austin’s

6. There are biscuits on the sideboard if you want some,

which sounds very odd indeed when phrased as an implication. Indeed, (6) poses perplexities of one kind or another for any theory of the conditional, and so should perhaps best be ignored as posing any special threat to the Anderson and Belnap account of conditionals. Perhaps it was Austin-type examples that led Anderson and Belnap [1975, pp. 491–492] to say “we
think every use of ‘implies’ or ‘entails’ as a connective can be replaced by a suitable ‘if-then’; however, the converse may not be true”. They go on to say “But with reference to the uses in which we are primarily interested, we feel free to move back and forth between ‘if-then’ and ‘entails’ in a free-wheeling manner”.

Associated with the Correspondence Thesis is the idea that just as there can be contingent conditionals (e.g. (1)), so then the corresponding implications (e.g. (3)) must also be contingent. This goes against certain Quinean tendencies to ‘regiment’ the English word ‘implies’ so that it stands only for logical implication. Although there is no objection to thus giving a technical usage to an ordinary English word (even requiring in this technical usage that ‘implication’ be a metalinguistic relation between sentences), the point is that relevance logicians by and large believe we are using ‘implies’ in the ordinary non-technical sense, in which a sentence like (3) might be true without there being any logical (or even necessary) implication from ‘Today is Tuesday’ to ‘This is Belgium’.

Relevance logicians are not themselves free of similar regimenting tendencies. Thus we tend to differentiate ‘entails’ from ‘implies’ on precisely the ground that ‘entails’, unlike ‘implies’, stands only for necessary implication [Meyer, 1966]. Some writings of Anderson and Belnap even suggest a more restricted usage for just logical implication, but we do not take this seriously. There does not seem to be any more linguistic evidence for thus restricting ‘entails’ than there would be for ‘implies’, though there may be at least more excuse given the apparently more technical history of ‘entails’ (in its logical sense—cf. The oed).

This has been an explanation of, if not an apology for, the ways in which relevance logicians often express themselves. but it should be stressed that the reader need not accept all, or any, of this background in order to make sense of the basic aims of the relevance logic enterprise. Thus, e.g. the reader may feel that, despite protestations to the contrary, Anderson, Belnap and Co. are hopelessly confused about the relationships among ‘entails’, ‘implies’, and ‘if-then’, but still think that their system R provides a good formalisation of the properties of ‘if-then’ (or at least ‘if-then relevantly’), and that they system E does the same for some strict variant produced by the modifier ‘necessarily’.

One of the reasons the recent logical tradition has been motivated to insist on the fierce distinction between implications and conditionals has to do with the awkwardness of reading the so-called ‘material conditional’ \( A \rightarrow B \) as corresponding to any kind of implication (cf. [Quine, 1953]).

The material conditional \( A \rightarrow B \) can of course be defined as \( \neg A \lor B \), and it certainly does seem odd, modifying an example that comes by oral tradition from Anderson, to say that:
7. Picking a guinea pig up by its tail implies that its eyes will fall out.

just on the grounds that its antecedent is false (since guinea pigs have no

tails). But then it seems equally false to say that:

8. If one picks up a guinea pig by its tail, then its eyes will fall out.

And also both of the following appear to be equally false:

9. Scaring a pregnant guinea pig implies that all of her babies will be

born tailless.

10. If one scares a pregnant guinea pig, then all of her babies will be born

tailless.

It should be noted that there are other ways to react to the oddity of sen-
tences like the ones above other than calling them simply false. Thus there

is the reaction stemming from the work of Grice [1975] that says that at

least the conditional sentences (8) and (10) above are true though nonethe-

less pragmatically odd in that they violate some rule based on conversa-
tional co-operation to the effect that one should normally say the strongest

thing relevant, i.e. in the cases above, that guinea pigs have no tails (cf. 


Also it should be noted that the theory of the ‘counterfactual’ conditional
due to Stalnaker-Thomason, D. K. Lewis and others (cf. Chapter [[??]] of 
this Handbook), while it agrees with relevance logic in finding sentences like

(8) (not (10) false, disagrees with relevance logic in the formal account it
gives of the conditional.

It would help matters if there were an extended discussion of these
competing theories (Anderson-Belnap, Grice, Stalnaker-Thomason-Lewis),
which seem to pass like ships in the night (can three ships do this without
strain to the image?) but there is not the space here. Such a discussion
might include an attempt to construct a theory of a relevant counterfactual
conditional (if A were to be the case, then as a result B would be the case).
The rough idea would be to use say The Routley-Meyer semantics for rele-
vance logic (cf. Section 3.7) in place of the Kripke semantics for modal
logic, which plays a key role in the Stalnaker-Thomason-Lewis semantical
account of the conditional (put the 3-placed alternativeness relation in the
role of the usual 2-placed one). Work in this area is just starting. See
the works of [Mares and Fuhrmann, 1995] and [Akama, 1997] which both
attempt to give semantics for relevant counterfactuals.

Also any discussion relating to Grice’s work would surely make much of
the fact that the theory of Grice makes much use of a basically unanalysed
notion of relevance. One of Grice’s chief conversational rules is ‘be relevant’,
but he does not say much about just what this means. One could look at
RELEVANCE LOGIC

7

relevance logic as trying to say something about this, at least in the case of
the conditional.

Incidentally, as Meyer has been at great pains to emphasise, relevance
logic gives, on its face anyway, no separate account of relevance. It is not
as if there is a unary relevance operator (‘relevantly’).

One last point, and then we shall turn to more substantive issues. Or-
thodox relevance logic differs from classical logic not just in having an ad-
ditional logical connective (→) for the conditional. If that was the only
difference relevance logic would just be an ‘extension’ of classical logic, us-
ing the notion of Haack [1974], in much the same way as say modal logic
is an extension of classical logic by the addition of a logical connective □
for necessity. The fact is (cf. Section 1.6) that although relevance logic
contains all the same theorems as classical logic in the classical vocabulary
say, ∧ ∨ ¬ (and the quantifiers), it nonetheless does not validate the same
inferences. Thus, most notoriously, the disjunctive syllogism (cf. Section 2)
is counted as invalid. Thus, as Wolf [1978] discusses, relevance logic does
not fit neatly into the classification system of [Haack, 1974], and might best
be called ‘quasi-extension’ of classical logic, and hence ‘quasi-deviant’. In-
cidentally, all of this applies only to ‘orthodox’ relevance logic, and not to
the ‘classical relevance logics’ of Meyer and Routley (cf. Section 3.11).

1.3 Hilbert-style Formulations

We shall discuss first the pure implicational fragments, since it is pri-
marily in the choice of these axioms that the relevance logics differ one
from the other. We shall follow the conventions of Anderson and Belnap
[Anderson and Belnap, 1975], denoting by ‘R−’, what might be called the
‘putative implicational fragment of R’. Thus R− will have as axioms all
the axioms of R that only involve the implication connective. That R−
is in fact the implicational fragment of R is much less than obvious since
the possibility exists that the proof of a pure implicational formula could
detour in an essential way through formulas involving connectives other
than implication. In fact Meyer has shown that this does not happen (cf.
his Section 28.3.2 of [Anderson and Belnap, 1975]), and indeed Meyer has
settled in almost every interesting case that the putative fragments of the
well-known relevance logics (at least R and E) are the same as the real
fragments. (Meyer also showed that this does not happen in one interesting
case, RM, which we shall discuss below.)

For R−, we take the rule modus ponens (A, A → B ⊢ B) and the following
axiom schemes.

\[
\begin{align*}
A & \rightarrow A \quad \text{Self-Implication} \quad (1) \\
(A \rightarrow B) & \rightarrow [(C \rightarrow A) \rightarrow (C \rightarrow B)] \quad \text{Prefixing} \quad (2) \\
\neg A \rightarrow (A \rightarrow B) & \rightarrow (A \rightarrow B) \quad \text{Contraction} \quad (3) \\
\neg A \rightarrow (B \rightarrow C) & \rightarrow [B \rightarrow (A \rightarrow C)] \quad \text{Permutation}. \quad (4)
\end{align*}
\]

A few comments are in order. This formulation is due to Church [1951b] who called it ‘The weak implication calculus’. He remarks that the axioms are the same as those of Hilbert’s for the positive implicational calculus (the implicational fragment of the intuitionistic propositional calculus \(H\)) except that (1) is replaced with

\[
A \rightarrow (B \rightarrow A) \quad \text{Positive Paradox.} \quad (1')
\]

(Recent historical investigation by Đošen [1992] has shown that Orlov constructed an axiomatisation of the implication and negation fragment of \(R\) in the mid 1920s, predating other known work in the area. Church and Moh, however, provided a Deduction Theorem (see Section 1.4) which is absent from Orlov’s treatment.)

The choice of the implicational axioms can be varied in a number of informative ways. Thus putting things quickly, (2) Prefixing may be replaced by

\[
(A \rightarrow B) \rightarrow [(B \rightarrow C) \rightarrow (A \rightarrow C)] \quad \text{Suffixing.} \quad (2')
\]

(3) Contraction may be replaced by

\[
\neg A \rightarrow (B \rightarrow C) \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)] \quad \text{Self-Distribution,} \quad (3')
\]

and (4) Permutation may be replaced by

\[
A \rightarrow [(A \rightarrow B) \rightarrow B] \quad \text{Assertion.} \quad (4')
\]

These choices of implicational axioms are ‘isolated’ in the sense that one choice does not affect another. Thus

THEOREM 1 \(R\), may be axiomatised with modus ponens, (1) Self-Implication and any selection of one from each pair \[\{(2), (2')\}, \{(3), (3')\}, \text{ and } \{(4), (4')\}\].

Proof. By consulting [Anderson and Belnap, 1975, pp. 79–80], and fiddling.

\[\blacksquare\]
There is at least one additional variant of $\mathbf{R}_\rightarrow$ that merits discussion. It turns out that it suffices to have Suffixing, Contraction, and the pair of axiom schemes

\[
[(A \rightarrow A) \rightarrow B] \rightarrow B \quad \text{Specialised Assertion,} \tag{4a}
\]
\[
A \rightarrow [(A \rightarrow A) \rightarrow A] \quad \text{Demodaliser.} \tag{4b}
\]

Thus (4b) is just an instance of Assertion, and (4a) follows from Assertion by substitution $A \rightarrow A$ for $A$ and using Self-Implication to detach. That (4a) and (4b) together with Suffixing and Contraction yield Assertion (and, less interestingly, Self-Implication) can be shown using the fact proven in [Anderson and Belnap, 1975, Section 8.3.3], that these yield (letting $\bar{A}$ abbreviate $A_1 \rightarrow A_2$)

\[
\bar{A} \rightarrow [(\bar{A} \rightarrow B) \rightarrow B] \quad \text{Restricted-Assertion.} \tag{4''}
\]

The point is that (4a) and (4b) in conjunction say that $A$ is equivalent to $(A \rightarrow A) \rightarrow A$, and so every formula $A$ has an equivalent form $\bar{A}$ and so ‘Restricted Assertion’ reduces to ordinary Assertion.²

Incidentally, no claim is made that this last variant of $\mathbf{R}_\rightarrow$ has the same isolation in its axioms as did the previous axiomatisations. Thus, e.g. that Suffixing (and not Prefixing) is an axiom is important (a matrix of J. R. Chidgey’s (cf. [Anderson and Belnap, 1975, Section 8.6]) can be used to show this.

The system $\mathbf{E}$ of entailment differs primarily from the system $\mathbf{R}$ in that it is a system of relevant strict implication. Thus $\mathbf{E}$ is both a relevance logic and a modal logic. Indeed, defining $\Box A =_d (A \rightarrow A) \rightarrow A$ one finds $\mathbf{E}$ has something like the modality structure of $\mathbf{S4}$ (cf. [Anderson and Belnap, 1975, Sections 4.3 and 10]).

This suggests that $\mathbf{E}_\rightarrow$, can be axiomatised by dropping Demodaliser from the axiomatisation of $\mathbf{R}_\rightarrow$, and indeed this is right (cf. [Anderson and Belnap, 1975, Section 8.3.3], for this and all other claims about axiomatisations of $\mathbf{E}_\rightarrow$).³

The axiomatisation above is a ‘fixed menu’ in that Prefixing cannot be replaced with Suffixing. There are other ‘à la carte’ axiomatisations in the style of Theorem 1.

**THEOREM 2** $\mathbf{E}_\rightarrow$ may be axiomatised with modus ponens, Self-Implication and any selection from each of the pairs \{Prefixing, Suffixing\}, \{Contraction, Self-Distribution\} and \{Restricted-Permutation, Restricted-Assertion\} (one from each pair).

²There are some subtleties here. Detailed analysis shows that both Suffixing and Prefixing are needed to replace $\bar{A}$ with $A$ (cf. Section 1.3). Prefixing can be derived from the above set of axioms (cf. [Anderson and Belnap, 1975, pp. 77–78 and p. 26]).

³The actual history is backwards to this, in that the system $\mathbf{R}$ was first axiomatised by [Belnap, 1967a] by adding Demodaliser to $\mathbf{E}$. 


Another implicational system of less central interest is that of ‘ticket entailment’ $\mathbf{T}_{\rightarrow}$. It is motivated by Anderson and Belnap [1975, Section 6] as deriving from some ideas of Ryle’s about ‘inference tickets’. It was motivated in [Anderson, 1960] as ‘entailment shorn of modality’. The thought behind this last is that there are two ways to remove the modal sting from the characteristic axiom of alethic modal logic, $\Box A \rightarrow A$. One way is to add Demodalisre $A \rightarrow \Box A$ so as to destroy all modal distinctions. The other is to drop the axiom $\Box A \rightarrow A$. Thus the essential way one gets $\mathbf{T}_{\rightarrow}$ from $\mathbf{E}_{\rightarrow}$ is to drop Specialised Assertion (or alternatively to drop Restricted Assertion or Restricted Permutation, depending on which axiomatisation of $\mathbf{E}_{\rightarrow}$ one has). But before doing so one must also add whichever one of Prefixing and Suffixing was lacking, since it will no longer be a theorem otherwise (this is easiest to visualise if one thinks of dropping Restricted permutation, since this is the key to getting Prefixing from Suffixing and vice versa). Also (and this is a strange technicality) one must replace Self-Distribution with its permuted form:

\[(A \rightarrow B) \rightarrow [[A \rightarrow (B \rightarrow C)] \rightarrow (A \rightarrow C)] \quad \text{Permuted Self-Distribution.} \tag{3''} \]

This is summarised in

**THEOREM 3** (Anderson and Belnap [Section 8.3.2, 1975]) $\mathbf{T}_{\rightarrow}$ is axiomatised using Self-Implication, Prefixing, Suffixing, and either of (Contraction, Permuted Self-Distribution), with modus ponens.

There is a subsystem of $\mathbf{E}_{\rightarrow}$ called $\mathbf{TW}_{\rightarrow}$ (and $\mathbf{P-W}$, and $\mathbf{T-W}$ in earlier nomenclature) axiomatised by dropping Contraction (which corresponds to the combinator $\mathbf{W}$) from $\mathbf{T}_{\rightarrow}$. This has obtained some interest because of an early conjecture of Belnap’s (cf. [Anderson and Belnap, 1975, Section 8.11]) that $A \rightarrow B$ and $B \rightarrow A$ are both theorems of $\mathbf{TW}_{\rightarrow}$ only when $A$ is the same formula as $B$. That Belnap’s Conjecture is now Belnap’s Theorem is due to the highly ingenious (and complicated) work of E. P. Martin and R. K. Meyer [1982] (based on the earlier work of L. Powers and R. Dwyer). Martin and Meyer’s work also highlights a system $\mathbf{S}_{\rightarrow}$ (for Syllogism) in which Self-Implication is dropped from $\mathbf{TW}_{\rightarrow}$.

Moving on now to adding the positive extensional connectives $\land$ and $\lor$, in order to obtain $\mathbf{R}_{\rightarrow, \land, \lor}$ (denoted more simply as $\mathbf{R}^+$) one adds to $\mathbf{R}_{\rightarrow}$,
the axiom schemes

\[
\begin{align*}
A \land B &\to A, ~ A \land B \to B & \text{Conjunction Elimination} \quad (5) \\
[(A \to B) \land (A \to C)] &\to (A \to B \land C) & \text{Conjunction Introduction} \quad (6) \\
A \to A \lor B, ~ B &\to A \lor B & \text{Disjunction Introduction} \quad (7) \\
[(A \to C) \land (B \to C)] &\to (A \lor B \to C) & \text{Disjunction Elimination} \quad (8) \\
A \land (B \lor C) &\to (A \land B) \lor C & \text{Distribution} \quad (9)
\end{align*}
\]

plus the rule of adjunction \((A,B \vdash A \land B)\). One can similarly get the positive intuitionistic logic by adding these all to \(H_\land\).

Axioms (5)–(8) can readily be seen to be encoding the usual elimination and introduction rules for conjunction and disjunction into axioms, giving \(\land\) and \(\lor\) what might be called ‘the lattice properties’ (cf. Section 3.3). It might be thought that \(A \to (B \to A \land B)\) might be a better encoding of conjunction introduction than (6), having the virtue that it allows for the dropping of adjunction. This is a familiar axiom for intuitionistic (and classical) logic, but as was seen by Church [1951b], it is only a hair’s breadth away from Positive Paradox \((A \to (B \to A))\), and indeed yields it given (5) and Prefixing. For some mysterious reason, this observation seemed to prevent Church from adding extensional conjunction/disjunction to what we now call \(R_\land\) (and yet the need for adjunction in the Lewis formulations of modal logic where the axioms are al strict implications was well-known).

Perhaps more surprising than the need for adjunction is the need for axiom (9). It would follow from the other axioms if only we had Positive Paradox among them. The place of Distribution in \(R\) is continually problematic. It causes inelegancies in the natural deduction systems (cf. Section 1.5) and is an obstacle to finding decision procedures (cf. Section 4.8). Incidentally, all of the usual distributive laws follow from the somewhat ‘clipped’ version (9).

The rough idea of axiomatising \(E_\lor^+\) and \(T_\lor^+\) is to add axiom schemes (5)–(9) to \(E_\land\) and \(T_\land\). This is in fact precisely right for \(T_\lor^+\), but for \(E_\lor^+\) one needs also the axiom scheme (remember \(\Box A =_d (A \to A) \to A\)):

\[
\Box A \land \Box B \to \Box(A \land B) \tag{10}
\]

This is frankly an inelegance (and one that strangely enough disappears in the natural deduction context of Section 1.5). It is needed for the inductive proof that necessitation \((\vdash C \Rightarrow \vdash \Box C)\) holds, handling the case where \(C\) just came by adjunction (cf. [Anderson and Belnap, 1975, Sections 21.2.2 and 23.4]). There are several ways of trying to conceal this inelegance, but they are all a little \textit{ad hoc}. Thus, e.g. one could just postulate the rule of necessitation as primitive, or one could strengthen the axiom of Restricted
Permuted (or Restricted Assertion) to allow that $\bar{A}$ be a conjunction $(A_1 \rightarrow A_2) \land (A_2 \rightarrow A_2)$.

As Anderson and Belnap [1975, Section 21.2.2] remark, if propositional quantification is available, $\Box A$ could be given the equivalent definition $\forall p(p \rightarrow p) \rightarrow A$, and then the offending (10) becomes just a special case of Conjunction Introduction and becomes redundant.

It is a good time to advertise that the usual zero-order and first-order relevance logics can be outfitted with a couple of optional convenience features that come with the higher-priced versions with propositional quantifiers. Thus, e.g. the propositional constant $t$ can be added to $\mathbf{E}^*$ to play the role of $\forall p(p \rightarrow p)$, governed by the axioms:

$$(t \rightarrow A) \rightarrow A \quad (11)$$

$$t \rightarrow (A \rightarrow A), \quad (12)$$

and again (10) becomes redundant (since one can easily show $(t \rightarrow A) \leftrightarrow [(A \rightarrow A) \rightarrow A]$).

Further, this addition of $t$ is conservative in the sense that it leads to no new $t$-free theorems (since in any given proof $t$ can always be replaced by $(p_1 \rightarrow p_1) \land \cdots \land (p_n \rightarrow p_n)$ where $p_1, \ldots, p_n$ are all the propositional variables appearing in the proof — cf. [Anderson and Belnap, 1975]).

Axiom scheme (11) is too strong for $\mathbf{T}^+$ and must be weakened to

$$t. \quad (11\mathbf{T})$$

In the context of $\mathbf{R}^+$, (11) and (11$\mathbf{T}$) are interchangeable, and in $\mathbf{R}^+$, (12) may of course be permuted, letting us characterise $t$ in a single axiom as ‘the conjunction of all truths’:

$$A \leftrightarrow (t \rightarrow A) \quad (13)$$

(in $\mathbf{E}$, $t$ may be thought of as ‘the conjunction of all necessary truths’).

‘Little $t$’ is distinguished from ‘big $T$’, which can be conservatively added with the axiom scheme

$$A \rightarrow T \quad (14)$$

(in intuitionistic or classical logic $t$ and $T$ are equivalent).

Additionally useful is a binary connective $\circ$, labelled variously ‘intensional conjunction’, ‘fusion’, ‘consistency’ and ‘cotenability’. These last two labels are appropriate only in the context of $\mathbf{R}$, where one can define $A \circ B = \text{df } \neg (A \rightarrow \neg B)$. One can add $\circ$ to $\mathbf{R}^+$ with the axiom scheme:

$$[(A \circ B) \rightarrow C] \leftrightarrow [A \rightarrow (B \rightarrow C)] \quad \text{Residuation (axiom).} \quad (15)$$
This axiom scheme is too strong for other standard relevance logics, but Meyer and Routley [1972] discovered that one can always add conservatively the two way rule
\[(A \circ B) \rightarrow C \vdash A \rightarrow (B \rightarrow C)\]  
Residuation (rule) \hspace{1cm} (16)

(in \(R^+\) (16) yields (15)). Before adding negation, we mention the positive fragment \(B^+\) of a kind of minimal (Basic) relevance logic due to Routley and Meyer (cf. Section 3.9). \(B^+\) is just like \(TW^+\) except for finding the axioms of Prefixing and Suffixing too strong and replacing them by rules:
\[A \rightarrow B \vdash (C \rightarrow A) \rightarrow (C \rightarrow B)\]  
Prefixing (rule) \hspace{1cm} (17)
\[A \rightarrow B \vdash (B \rightarrow C) \rightarrow (A \rightarrow C)\]  
Suffixing (rule) \hspace{1cm} (18)

As for negation, the full systems \(R, E, \) etc. may be formed adding to the axiom schemes for \(R^+, E^+, \) etc. the following \(^4\)
\[
(A \rightarrow \neg A) \rightarrow \neg A \quad \text{Reductio} \hspace{1cm} (19)
\]
\[
(A \rightarrow \neg B) \rightarrow (B \rightarrow \neg A) \quad \text{Contraposition} \hspace{1cm} (20)
\]
\[
\neg \neg A \rightarrow A \quad \text{Double Negation.} \hspace{1cm} (21)
\]

Axiom schemes (19) and (20) are intuitionistically acceptable negation principles, but using (21) one can derive forms of reductio and contraposition that are intuitionistically rejectable. Note that (19)–(21) if added to \(H^+\) would give the full intuitionistic propositional calculus \(H.\)

In \(R,\) negation can alternatively be defined in the style of Johansson, with \(\neg A \equiv_{df} (A \rightarrow f),\) where \(f\) is a false propositional constant, cf. [Meyer, 1966]. Informally, \(f\) is the disjunction of all false propositions (the ‘negation’ of \(t\)). Defining negation thus, axiom schemes (19) and (20) become theorems (being instances of Contraction and Permutation, respectively). But scheme (21) must still be taken as an axiom.

Before going on to discuss quantification, we briefly mention a couple of other systems of interest in the literature.

Given that \(E\) has a theory of necessity riding piggyback on it in the definition \(\Box A \equiv_{df} (A \rightarrow A) \rightarrow A,\) the idea occurred to Meyer of adding to \(R\) a primitive symbol for necessity \(\Box\) governed by the \(S4\) axioms.

\[
\Box A \rightarrow A \hspace{1cm} (\Box 1)
\]
\[
\Box (A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B) \hspace{1cm} (\Box 2)
\]
\[
\Box A \land \Box B \rightarrow \Box (A \land B) \hspace{1cm} (\Box 3)
\]
\[
\Box A \rightarrow \Box \Box A, \hspace{1cm} (\Box 4)
\]

\(^4\)Reversing what is customary in the literature, we use \(\neg\) for the standard negation of relevance logic, reserving \(\sim\) for the ‘Boolean negation’ discussed in Section 3.11. We do this so as to follow the notational policies of the \textit{Handbook}.\)
and the rule of Necessitation (\( \vdash A \Rightarrow \vdash \Box A \)).

His thought was that \( E \) could be exactly translated into this system \( R^2 \) with entailment defined as strict implication. That this is subtly not the case was shown by Maksimova [1973] and Meyer [1979b] has shown how to modify \( R^2 \) so as to allow for an exact translation.

Yet one more system of interest is \( RM \) (cf. Section 3.10) obtained by adding to \( R \) the axiom scheme

\[
A \rightarrow (A \rightarrow A) \quad \text{Mingle.} \tag{22}
\]

Meyer has shown somewhat surprisingly that the pure implicational system obtained by adding Mingle to \( R \) is not the implicational fragment of \( RM \), and he and Parks have shown how to axiomatise this fragment using a quite unintelligible formula (cf. [Anderson and Belnap, 1975, Section 8.18]). Mingle may be replaced equivalently with the converse of Contraction:

\[
(A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B)) \quad \text{Expansion.} \tag{23}
\]

Of course one can consider ‘mingled’ versions of \( E \), and indeed it was in this context that McCall first introduced mingle, albeit in the strict form (remember \( \bar{A} = A_1 \rightarrow A_2 \)),

\[
\bar{A} \rightarrow (\bar{A} \rightarrow \bar{A}) \quad \text{Mingle} \tag{24}
\]

(cf. [Dunn, 1976c]).

We finish our discussion of axiomatics with a brief discussion of first-order relevance logics, which we shall denote by \( RQ, EQ \), etc. We shall presuppose a standard definition of first-order formula (with connectives \( \neg, \land, \lor, \rightarrow \) and quantifiers \( \forall, \exists \)). For convenience we shall suppose that we have two denumerable stocks of variables: the bound variables \( x, y, \) etc. and the free variables (sometimes called parameters) \( a, b, \) etc. The bound variables are never allowed to have unbound occurrences.

The quantifier laws were set down by Anderson and Belnap in accord with the analogy of the universal quantifier with a conjunction (or its instances), and the existential quantifier as a disjunction. In view of the validity of quantifier interchange principles, we shall for brevity take only the universal quantifier \( \forall \) as primitive, defining \( \exists x A =_{df} \neg \forall x \neg A \). We thus need

\[
\forall x A \rightarrow A(a/x) \quad \forall\text{-elimination} \tag{25}
\]

\[
\forall x (A \rightarrow B) \rightarrow (A \rightarrow \forall x B) \quad \forall\text{- introduction} \tag{26}
\]

\[
\forall x (A \lor B) \rightarrow A \lor \forall x B \quad \text{Confinement.} \tag{27}
\]

If there are function letters or other term forming operators, then (25) should be generalised to \( \forall x A \rightarrow A(t/x) \), where \( t \) is any term (subject to
our conventions that the ‘bound variables’ $x, y$, etc. do not occur (‘free’) in it). Note well that because of our convention that ‘bound variables’ do not occur free, the usual proviso that $x$ does not occur free in $A$ in (26) and (27) is automatically satisfied. (27) is the obvious ‘infinite’ analogy of Distribution, and as such it causes as many technical problems for $\mathbf{RQ}$ as does Distribution for $\mathbf{R}$ (cf. Section 4.8). Finally, as an additional rule corresponding to adjunction, we need:

$$\frac{A(a/x)}{\forall x A} \quad \text{Generalisation.} \quad (28)$$

There are various more or less standard ways of varying this formulation. Thus, e.g. (cf. Meyer, Dunn and Leblanc [1974]) one can take all universal generalisations of axioms, thus avoiding the need for the rule of Generalisation. Also (26) can be ‘split’ into two parts:

$$\forall x(A \rightarrow B) \rightarrow (\forall x A \rightarrow \forall x B)$$

$$A \rightarrow \forall x A \quad \text{Vacuous Quantification} \quad (26a)$$

$$A 

(again note that if we allowed $x$ to occur free we would have to require that $x$ not be free in $A$).

The most economical formulation is due to Meyer [1970]. It uses only the axiom scheme of $\forall$-elimination and the rule.

$$\frac{A \rightarrow B \lor C(a/x)}{A \rightarrow B \lor \forall x C} \quad (a \text{ cannot occur in } A \text{ or } B) \quad (29)$$

which combines (26)–(28).

### 1.4 Deduction Theorems in Relevance Logic

Let $\mathbf{X}$ be a formal system, with certain formulas of $\mathbf{X}$ picked out as axioms and certain (finitary) relations among the formulas of $\mathbf{X}$ picked out as rules. (For the sake of concreteness, $\mathbf{X}$ can be thought of as any of the Hilbert-style systems of the previous section.) Where $\Gamma$ is a list of formulas of $\mathbf{X}$ (thought of as hypotheses) it is customary to define a deduction from $\Gamma$ to be a sequence $B_1, \ldots, B_n$, where for each $B_i (1 \leq i \leq n)$, either (1) $B_i$ is in $\Gamma$, or (2) $B_i$ is an axiom of $\mathbf{X}$, or (3) $B_i$ ‘follows from’ earlier members of the sequence, i.e. $R(B_{j_1}, \ldots, B_{j_k}, B_i)$ holds for some $(k + 1)$—any rule of $\mathbf{X}$ and $B_{j_1}, \ldots, B_{j_k}$ all precede $B_i$ in the sequence $B_1, \ldots, B_n$. A formula $A$ is then said to be deducible from $\Gamma$ just in case there is some deduction from $\Gamma$ terminating in $A$. We symbolise this as $\Gamma \vdash_{\mathbf{X}} A$ (often suppressing the subscript).

A proof is of course a deduction from the empty set, and a theorem is just the last item in a proof. There is the well-known
Deduction Theorem (Herbrand). If \( A_1, \ldots, A_n, A \vdash_{\mathbf{H}_\rightarrow} B \), then we have also \( A_1, \ldots, A_n \vdash_{\mathbf{H}_\rightarrow} A \rightarrow B \).

This theorem is proven in standard textbooks for classical logic, but the standard inductive proof shows that in fact the Deduction Theorem holds for any formal system \( \mathbf{X} \) having \textit{modus ponens} as its sole rule and \( \mathbf{H}_\rightarrow \subseteq \mathbf{X} \) (i.e. each instance of an axiom scheme of \( \mathbf{H}_\rightarrow \) is a theorem of \( \mathbf{X} \)). Indeed \( \mathbf{H}_\rightarrow \) can be motivated as the minimal pure implicational calculus having modus ponens as its sole rule and satisfying the Deduction Theorem. This is because the axioms of \( \mathbf{H}_\rightarrow \) can all be derived as theorems in any formal system \( \mathbf{X} \) using merely \textit{modus ponens} and the supposition that \( \mathbf{X} \) satisfies the Deduction Theorem. Thus consider as an example:

\[
\begin{align*}
(1) & \quad A, B \vdash A & \text{Definition of } \vdash \\
(2) & \quad A \vdash B \rightarrow A & \text{(1), Deduction Theorem} \\
(3) & \quad \vdash A \rightarrow (B \rightarrow A) & \text{(2), Deduction Theorem}.
\end{align*}
\]

Thus the most problematic axiom of \( \mathbf{H}_\rightarrow \) has a simple ‘\textit{a priori} deduction’, indeed one using only the Deduction Theorem, not even \textit{modus ponens} (which is though needed for more sane axioms like Self-Distribution).

It might be thought that the above considerations provide a very powerful argument for motivating intuitionistic logic (or at least some logic having the same implicational fragment) as The One True Logic. For what else should an implication do but satisfy \textit{modus ponens} and the Deduction Theorem?

But it turns out that there is another sensible notion of deduction. This is what is sometimes called a \textit{relevant deduction.} (Anderson and Belnap [1975, Section 22.2.1] claim that this is the \textit{only} sensible notion of deduction, but we need not follow them in that). If there is anything that sticks out in the \textit{a priori} deduction of Positive Paradox above it is that in (1), \( B \) was not \textit{used} in the deduction of \( A \).

A number of researchers have been independently bothered by this point and have been motivated to study a relevant implication that goes hand in hand with a notion of relevant deduction. This, in this manner Moh [1950] and Church [1951b] came up with what is in effect \( \mathbf{R}_\rightarrow \). And Anderson and Belnap [1975, p. 261] say “In fact, the search for a suitable deduction theorem for Ackermann’s systems \ldots provided the impetus leading us to the research reported in this book.” This research program begun in the late 1950s took its starting point in the system(s) of Ackermann [1956], and the bold stroke separating the Anderson–Belnap system \( \mathbf{E} \) from Ackermann’s system II’ was basically the dropping of Ackermann’s rule \( \gamma \) so as to have an appropriate deduction theorem (cf. Section 2.1).

Let us accordingly define a deduction of \( B \) from \( A_1, \ldots, A_n \) to be \textit{relevant with respect to a given hypothesis} \( A_i \) just in case \( A_i \) is actually \textit{used} in the given deduction of \( B \) in the sense (paraphrasing [Church, 1951b]) that
there is a chain of inferences connecting $A_i$ with the final formula $B$. This last can be made formally precise in any number of ways, but perhaps the most convenient is to flag $A_i$ with say a $\dagger$ and to pass the flag along in the deduction each time modus ponens is applied to two items at least one of which is flagged. It is then simply required that the last step of the deduction ($B$) be flagged. Such devices are familiar from various textbook presentations of classical predicate calculus when one wants to keep track whether some hypothesis $A_i(x)$ was used in the deduction of some formula $B(x)$ to which one wants to apply Universal Generalisation.

We shall define a deduction of $B$ from $A_1, \ldots, A_n$ to be relevant simpliciter just in case it is relevant with respect to each hypothesis $A_i$. A practical way to test for this is to flag each $A_i$ with a different flag (say the subscript $i$) and then demand that all of the flags show up on the last step $B$.

We can now state a version of the

**Relevant Deduction Theorem (Moh, Church).** *If there is a deduction in $R_\infty$ of $B$ from $A_1, \ldots, A_n$, $A$ that is relevant with respect to $A$, then there is a deduction in $R_\infty$ of $A \rightarrow B$ from $A_1, \ldots, A_n$. Furthermore the new deduction will be ‘as relevant’ as the old one, i.e. any $A_i$ that was used in the given deduction will be used in the new deduction.*

**Proof.** Let the given deduction be $B_1, \ldots, B_k$, and let it be given with a particular analysis as to how each step is justified. By induction we show for each $B_i$ that if $A$ was used in obtaining $B_i$ ($B_i$ is flagged), then there is a deduction of $A \rightarrow B_i$ from $A_1, \ldots, A_n$, and otherwise there is a deduction of $B_i$ from those same hypotheses. The tedious business of checking that the new deduction is as relevant as the old one is left to the reader. We divide up cases depending on how the step $B_i$ is justified.

**Case 1.** $B_i$ was justified as a hypothesis. Then neither $B_i$ is $A$ or it is some $A_j$. But $A \rightarrow A$ is an axiom of $R_\infty$ (and hence deducible from $A_1, \ldots, A_n$), which takes care of the first alternative. And clearly on the second alternative $B_i$ is deducible from $A_1, \ldots, A_n$ (being one of them).

**Case 2.** $B_i$ was justified as an axiom. Then $A$ was not used in obtaining $B_i$, and of course $B_i$ is deducible (being an axiom).

**Case 3.** $B_i$ was justified as coming from preceding steps $B_j \rightarrow B_i$ and $B_j$ by modus ponens. There are four subcases depending on whether $A$ was used in obtaining the premises.

**Subcase 3.1.** $A$ was used in obtaining both $B_j \rightarrow B_i$ and $B_j$. Then by inductive hypothesis $A_1, \ldots, A_n \vdash_{R_\infty} A \rightarrow (B_j \rightarrow B_i)$ and $A_1, \ldots, A_n \vdash_{R_\infty} A \rightarrow B_j$. So $A \rightarrow B$ may be obtained using the axiom of Self-Distribution.

**Subcase 3.2.** $A$ was used in obtaining $B_j \rightarrow B_i$ but not $B_j$. Use the axiom of Permutation to obtain $A \rightarrow B_i$ from $A \rightarrow (B_j \rightarrow B_i)$ and $B_j$. 

Subcase 3.3. A was not used in obtaining $B_j \rightarrow B_i$ but was used for $B_j$. Use the axiom of Prefixing to obtain $A \rightarrow B_i$ from $B_j \rightarrow B_i$ and $A \rightarrow B_j$.

Subcase 3.4. A was not used in obtaining either $B_j \rightarrow B_i$ nor $B_j$. Then $B_i$ follows from these using just modus ponens.

Incidentally, $\mathbf{R}_- -$ can easily be verified to be the minimal pure implicational calculus having modus ponens as sole rule and satisfying the Relevant Deduction Theorem, since each of the axioms invoked in the proof of this theorem can be easily seen to be theorems in any such system (cf. the next section for an illustration of sorts).

There thus seem to be at least two natural competing pure implicational logics $\mathbf{R}_-$, and $\mathbf{H}_-$, differing only in whether one wants one’s deductions to be relevant or not.\footnote{This seems to differ from the good-humoured polemical stand of Anderson and Belnap [1975, Section 22.2.1], which says that the first kind of ‘deduction’, which they call (pejoratively) ‘Official deduction’, is no kind of deduction at all.}

Where does the Anderson–Belnap’s [1975] preferred system $\mathbf{E}_-$ fit into all of this? The key is that the implication of $\mathbf{E}_-$ is both a strict and a relevant implication (cf. Section 1.3 for some subtleties related to this claim). As such, and since Anderson and Belnap have seen fit to give it the modal structure of the Lewis system $\mathbf{S}_4$, it is appropriate to recall the appropriate deduction theorem for $\mathbf{S}_4$.

**Modal Deduction Theorem** [Barcan Marcus, 1946] If $A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n, A \vdash_{\mathbf{S}_4} B$ (← here denotes strict implication), then $A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n \vdash_{\mathbf{S}_4} A \rightarrow B$.

The idea here is that in general in order to derive the strict (necessary) implication $A \rightarrow B$ one must not only be able to deduce $B$ from $A$ and some other hypotheses but furthermore those other hypotheses must be supposed to be necessary. And in $\mathbf{S}_4$ since $A_i \rightarrow B_j$ is equivalent to $\Box(A_i \rightarrow B_j)$, requiring those additional hypotheses to be strict implications at least suffices for this.

Thus we could only hope that $\mathbf{E}_-$ would satisfy the

**Modal Relevant Deduction Theorem** [Anderson and Belnap, 1975] If there is a deduction in $\mathbf{E}_-$ of $B$ from $A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n, A$ that is relevant with respect to $A$, then there is a deduction in $\mathbf{E}_-$ of $A \rightarrow B$ from $A_1 \rightarrow B_1, \ldots, A_n \rightarrow B_n$ that is as relevant as the original.

The proof of this theorem is somewhat more complicated than its unmodalised counterpart which we just proved (cf. [Anderson and Belnap, 1975, Section 4.21] for a proof).

We now examine a subtle distinction (stressed by Meyer—see, for example, [Anderson and Belnap, 1975, pp. 394–395]), postponed until now for
pedagogical reasons. We must ask, how many hypotheses can dance on the head of a formula? The question is: given the list of hypotheses \( A, A \), do we have one hypothesis or two? When the notion of a \textit{deduction} was first introduced in this section and a \textquote{list} of hypotheses \( \Gamma \) was mentioned, the reader would naturally think that this was just informal language for a set. And of course the set \( \{ A, A \} \) is identical to the set \( \{ A \} \). Clearly \( A \) is relevantly deducible from \( A \). The question is whether it is so deducible from \( A, A \). We have then two different criteria of use, depending on whether we interpret hypotheses as grouped together into lists that distinguish multiplicity of occurrences (sequences)\(^6\) or sets. This issue has been taken up elsewhere of late, with other accounts of deduction appealing to \textquote{resource consciousness} [Girard, 1987, Troelstra, 1992, Schroeder-Heister and Došen, 1993] as motivating some non-classical logics. Substructural logics in general appeal to the notion that the number of times a premise is used, or even more radically, the \textit{order} in which premises are used, matter.

At issue in \( R \) and its neighbours is whether \( A \to (A \to A) \) is a correct relevant implication (coming by two applications of \textquote{The Deduction Theorem} from \( A, A \vdash A \)). This is in fact not a theorem of \( R \), but it is the characteristic axiom of \( \text{RM} \) (cf. Section 1.3). So it is important that in the Relevant Deduction Theorem proved for \( R \), that the hypotheses \( A_1, \ldots, A_n \) be understood as a sequence in which the same formula may occur more than once. One can prove a version of the Relevant Deduction Theorem with hypotheses understood as collected into a set for the system \( \text{RMO} \), obtained by adding \( A \to (A \to A) \) to \( R \) (but the reader should be told that Meyer has shown that \( \text{RMO} \) is \textit{not} the implicational fragment of \( \text{RM} \), cf. [Anderson and Belnap, 1975, Section 8.15]).\(^7\)

Another consideration pointing to the \textit{naturalness} of \( R \) is its connection to the \( \lambda I \)-calculus. A formula is a theorem of \( R \) if and only if it is the type of a closed term of the \( \lambda I \)-calculus as defined by Church. A \( \lambda I \) term is a \( \lambda \) term in which every lambda abstraction binds at least one free variable. So, \( \lambda x.\lambda y.x \) has type \( A \to ((A \to B) \to B) \), and so, \( A \) is a theorem of \( R \), while \( \lambda x.\lambda y.x \) has type \( A \to (B \to A) \), which is an intuitionistic theorem, but not an \( R \) theorem. This is reflected in the \( \lambda \) term, in which the \( \lambda y \) does not bind a free variable.

We now briefly discuss what happens to deduction theorems when the

---

\(^6\)Sequences are not quite the best mathematical structures to represent this grouping since it is clear that the order of hypotheses makes no difference (at least in the case of \( R \)). Meyer and McRobbie [1979] have investigated \textquote{firesets} (finitely repeatable sets) as the most appropriate abstraction.

\(^7\)Avron has defended this system, \( \text{RMO} \), as a natural way to characterise relevant implication [Avron, 1986, Avron, 1990b, Avron, 1990c, Avron, 1992]. In Avron’s system, conjunction and disjunction are \textit{intensional} connectives, defined in terms of the implication and negation of \( \text{RMO} \). As a result, they do not have all of the distributive lattice properties of traditional relevance logics.
pure implication systems $\text{R}_{\ldots}$ and $\text{E}_{\ldots}$ are extended to include other connectives, especially $\wedge$. $\text{R}$ will be the paradigm, its situation extending straightforwardly to $\text{E}$. The problem is that the full system $\text{R}$ seems not to be formulable with modus ponens as the sole rule; there is also need for adjunction $(A, B \vdash A \wedge B)$ (cf. Section 1.3).

Thus when we think of proving a version of the Relevant Deduction Theorem for the full system $\text{R}$, it would seem that we are forced to think through once more the issue of when a hypothesis is used, this time with relation to adjunction. It might be thought that the thing to do would be to pass the flag $\sharp$ along over an application of adjunction so that $A \wedge B$ ends up flagged if either of the premises $A$ or $B$ was flagged, in obvious analogy with the decision concerning modus ponens.

Unfortunately, that decision leads to disaster. For then the deduction $A, B \vdash A \wedge B$ would be a relevant one (both $A$ and $B$ would be ‘used’), and two applications of ‘The Deduction Theorem’ would lead to the thesis $A \rightarrow (B \rightarrow A \wedge B)$, the undesirability of which has already been remarked.

A more appropriate decision is to count hypotheses as used in obtaining $A \wedge B$ just when they were used to obtain both premises. This corresponds to the axiom of Conjunction Introduction $(C \rightarrow A) \wedge (C \rightarrow B) \rightarrow (C \rightarrow A \wedge B)$, which thus handles the case in the inductive proof of the deduction theorem when the adjunction is applied. This decision may seem ad hoc (perhaps ‘use’ simpliciter is not quite the right concept), but it is the only decision to be made unless one wants to say that the hypothesis $A$ can (in the presence of the hypothesis $B$) be ‘used’ to obtain $A \wedge B$ and hence $B$ (passing on the flag from $A$ this way is something like laundering dirty money).

This is the decision that was made by Anderson and Belnap in the context of natural deduction systems (see next section), and it was applied by Kron [1973, 1976] in proving appropriate deduction theorems for $\text{R}$, $\text{E}$ (and $\text{T}$). It should be said that the appropriate Deduction Theorem requires simultaneous flagging of the hypothesis (distinct flags being applied to each formula occurrence, say using subscripts in the manner of the ‘practical suggestion’ after our definition of relevant deduction for $\text{R}_{\ldots}$), with the requirement that all of the subscripts are passed on to the conclusion. So the Deduction Theorem applies only to fully relevant deductions, where every premise is used (note that no such restriction was placed on the Relevant Deduction Theorem for $\text{R}_{\ldots}$).

An alternative stated in Meyer and McRobbie [1979] would be to adjust the definition of deduction, modifying clause (2) so as to allow as a step in a deduction any theorem (not just axiom) of $\text{R}$, and to restrict clause (3) so that the only rule allowed in moving to later steps is modus ponens.\textsuperscript{*} This

\textsuperscript{*}Of course this requires we give an independent characterisation of proof (and theorem), since we can no longer define a proof as a deduction from zero premisses. We thus
is in effect to restrict adjunction to theorems, and reminds one of similar restrictions in the context of deduction theorems of similarly restricting the rules of necessitation and universal generalisation. It has the virtue that the Relevant Deduction Theorem and its proof are the same as for $R$. (Incidentally, Meyer’s and Kron’s sense of deduction coincide when all of $A_1, \ldots, A_n$ are used in deducing $B$; this is obvious in one direction, and less than obvious in the other.)

There are yet two other versions of the deduction theorem that merit discussion in the context of relevance logic (relevance logic, as Meyer often points out, allows for many distinctions).

First in Belnap [1960b] and Anderson and Belnap [1975], there is a theorem (stated for $E$, but we will state it for our paradigm $R$) called The Entailment Theorem, which says that $A_1, \ldots, A_n$ ‘entails’ $B$ iff $\vdash_R (A_1 \land \ldots \land A_n) \rightarrow B$. A formula $B$ is defined in effect to be entailed by hypothesis $A_1, \ldots, A_n$ just in case there is a deduction of $B$ using their conjunction $A_1 \land \ldots \land A_n$. Adjunction is allowed, but subject to the restriction that the conjunctive hypothesis was used in obtaining both premises. The Entailment Theorem is clearly implied by Kron’s version of the Deduction Theorem.

The last deduction theorem for $R$ we wish to discuss is the

ENTHYPHOMATIC DEDUCTION THEOREM (Meyer, Dunn and Leblanc [1974]).

If $A_1, \ldots, A_n \vdash_R B$, then $A_1, \ldots, A_n \vdash_R A \land t \rightarrow B$.

Here ordinary deducibility is all that is at issue (no insistence on the hypotheses being used). It can either be proved by induction, or cranked out of one of the more relevant versions of the deduction theorem. Thus it falls out of the Entailment Theorem that

$$\vdash_R X \land A \land T \rightarrow B,$$

where $X$ is the conjunction of $A_1, \ldots, A_n$, and $T$ is the conjunction of all the axioms of $R$ used in the deduction of $B$. But since $\vdash_R t \rightarrow T$, we have $\vdash_R X \land A \land t \rightarrow B$.

However, the following $R$ theorem holds:

$$\vdash_R (X \land A \land t \rightarrow B) \rightarrow (X \land t \rightarrow (A \land t \rightarrow B)).$$

So $\vdash_R X \land t \rightarrow (A \land t \rightarrow B)$, which leads (using $\vdash_R t$) to $X \vdash_R A \land t \rightarrow B$, which dissolving the conjunction gives the desired

$$A_1, \ldots, A_n \vdash_R A \land t \rightarrow B.$$

define a proof as a sequence of formulas, each of which is either an axiom or follows from preceding items by either modus ponens or adjunction (!).
In view of the importance of the notion, let us symbolise \( A \land t \rightarrow B \) as \( A \rightarrow_t B \). This functions as a kind of ‘enthemmatic implication’ (\( A \) and some truth really implies \( B \)) and there will be more about Anderson, Belnap and Meyer’s investigations of this concept in Section 1.7. Let us simply note now that in the context of deduction theorems, it functions like intuitionistic implication, and allows us in \( R \) to have two different kinds of implication, each well motivated in its relation to the two different kinds of deductibility (ordinary and relevant).\(^9\) For a more extensive discussion of deduction theorems in relevance logics and related systems, more recent papers by Avron [1991] and Brady [1994] should be consulted.

1.5 Natural Deduction Formulations

We shall be very brief about these since natural deduction methods are amply discussed by Anderson and Belnap [1975], where such methods in fact are used s a major motivation for relevance logic. Here we shall concentrate on a natural deduction system \( NR \) for \( R \).

The main idea of natural deduction (cf. Chapters [I.1 and I.2]) of the Handbook of course is to allow the making of temporary hypotheses, with some device usually being provided to facilitate the book-keeping concerning the use of hypotheses (and when their use is ‘discharged’). Several textbooks (for example, [Suppes, 1957] and [Lenmon, 1965])\(^10\) have used the device of in effect subscripting each hypothesis made with a distinct numeral, and then passing this numeral along with each application of a rule, thus keeping track of which hypothesis are used. When a hypothesis is discharged, the subscript is dropped. A line obtained with no subscripts is a ‘theorem’ since it depends on no hypotheses.

Let us then let \( \alpha, \beta, \) etc. range over classes of numerals. The rules for \( \rightarrow \) are then naturally:

\[
\begin{align*}
A \rightarrow B_\alpha & \quad \frac{A \rightarrow B_\beta}{B_{\alpha} \rightarrow B_{\beta}} & \\
A_\beta & \quad \frac{A_\beta}{B_{\alpha} \rightarrow B_{\beta}} \quad \vdots \\
& \quad \frac{B_{\alpha}}{A \rightarrow B_{\alpha - (k)}} \quad (\text{provided } k \in \alpha)
\end{align*}
\]

Two fussy, really incidental remarks must be made. First, in the rule \( \rightarrow E \) it is to be understood that the premises need not occur in the order listed, nor need they be adjacent to each other or to the conclusion. Otherwise we would need a rule of ‘Repetition’, which allows the repeating of a formula with its subscripts as a later line. (Repetition is trivially derivable given

\(^9\)In \( E \) enthematic implication is like \( S4 \) strict implication. See [Meyer, 1970a].

\(^10\)The idea actually originates with [Peys and Ladrière, 1955].
our ‘non-adjacent’ understanding of $\rightarrow E$—in order to repeat $A_\alpha$, just prove $A \rightarrow A$ and apply $\rightarrow E$.) Second, it is understood that we have what one might call a rule of ‘Hypothesis Introduction’: anytime one likes one can write a formula as a line with a new subscript (perhaps most conveniently, the line number).

Now a non-fussy remark must be made, which is really the heart of the whole matter. In the rule for $\rightarrow I$, a proviso has been attached which has the effect of requiring that the hypothesis $A$ was actually used in obtaining $B$. This is precisely what makes the implication relevant (one gets the intuitionistic implication system $\mathbf{H}$ if one drops this requirement). The reader should find it instructive to attempt a proof of Positive Paradox $(A \rightarrow (B \rightarrow A))$ and see how it breaks down for $\mathbf{NR}_\rightarrow$ (but succeeds in $\mathbf{NH}_\rightarrow$. The reader should also construct proofs in $\mathbf{NR}_\rightarrow$ of all the axioms in one of the Hilbert-style formulations of $\mathbf{R}_\rightarrow$, from Section 1.3.

Then the equivalence of $\mathbf{R}_\rightarrow$ in its Hilbert-style and natural deduction formulations is more or less self-evident given the Relevant Deduction Theorem (which shows that the rule $\rightarrow I$ can be ‘simulated’ in the Hilbert-style system, the only point at issue).

Indeed it is interesting to note that Lemmon [1965], who seems to have the same proviso on $\rightarrow I$ that we have for $\mathbf{NR}_\rightarrow$ (his actual language is a bit informal), does not prove Positive Paradox until his second chapter adding conjunction (and disjunction) to the implication-negation system he developed in his first chapter. His proof of Positive Paradox depends finally upon an ‘irrelevant’ $\rightarrow I$ rule. The following is perhaps the most straightforward proof in his system (differing from the proof he actually gives):

$$
\begin{align*}
(1) & \quad A_1 & \text{Hyp} \\
(2) & \quad B_2 & \text{Hyp} \\
(3) & \quad A \land B_{1,2} & 1, 2, \land I! \\
(4) & \quad A_{1,2} & 3, \land E \\
(5) & \quad B \rightarrow A_1 & 2, 4, \rightarrow I \\
(6) & \quad A \rightarrow (B \rightarrow A) & 1, 5, \rightarrow I.
\end{align*}
$$

We think that the manoeuvre used in getting $B$’s 2 to show up attached to $A$ in line (4) should be compared to laundering dirty money by running it through an apparently legitimate business. The correct ‘relevant’ versions of the conjunction rules are instead

$$
\begin{align*}
& \quad \frac{A_\alpha}{A \land B_\alpha} \quad [\land I] \\
& \quad \frac{A \land B_\alpha}{A_\alpha} \quad \frac{A \land B_\alpha}{B_\alpha} \quad \frac{A \land B_\alpha}{B_\alpha} \quad [\land E]
\end{align*}
$$

What about disjunction? In $\mathbf{R}$ (also $\mathbf{E}$, etc.) one has de Morgan’s Laws and Double Negation, so one can simply define $A \lor B = \neg \neg (A \land \neg B)$. One might
think that settling down in separate int-elim rules for $\vee$ would then only be a matter of convenience. Indeed, one can find in [Anderson and Belnap, 1975] in effect the following rules:

\[
\frac{A \vee B_\alpha}{A \vee B_\alpha} \quad \vdots \quad \frac{A_k}{A_k} \\
\frac{B_\alpha}{A \vee B_\alpha} \quad \vdots \quad \frac{B_h}{B_h} \\
\frac{\left[ \vee I \right]}{C_{\beta,\gamma}(k)} \quad \frac{\left[ \vee E \right]}{C_{\beta,\gamma}(h)}
\]

But (as Anderson and Belnap point out) these rules are insufficient. From them one cannot derive the following

\[
\frac{A \land (B \vee C)_\alpha}{(A \land B) \lor C_\alpha} \quad \text{Distribution.}
\]

And so it must be taken as an additional rule (even if disjunction is defined from conjunction and negation).

This is clearly an unsatisfying, if not unsatisfactory, state of affairs. The customary motivation behind int-elim rules is that they show how a connective may be introduced into and eliminated from argumentative discourse (in which it has no essential occurrence), and thereby give the connective’s rule or meaning. In this context the Distribution rule looks very much to be regretted.

One remedy is to modify the natural deduction system by allowing hypotheses to be introduced in two different ways, ‘relevantly’ and ‘irrelevantly’. The first way is already familiar to us and is what requires a subscript to keep track of the relevance of the hypothesis. It requires that the hypotheses introduced this way will all be used to get the conclusion. The second way involves only the weaker promise that at least some of the hypotheses so introduced will be used. This suggestion can be formalised by allowing several hypotheses to be listed on a line, but with a single relevance numeral attached to them as a bunch. Thus, schematically, an argument of the form

\[
(1) \quad A, B_1 \\
(2) \quad C, D_2 \\
\quad \vdots \\
(k) \quad E_{1,2}
\]
should be interpreted as establishing
\[ A \land B \rightarrow (C \land D \rightarrow E). \]

Now the natural deduction rules must be stated in a more general form allowing for the fact that more than one formula can occur on a line. Key among these would be the new rule:

\[
\Gamma, A \lor B_\alpha \\
\vdots \\
\Gamma, A_k \\
\vdots \\
\Delta_{\alpha \cup \beta} [\lor'] \\
\Gamma, B_\beta \\
\vdots \\
\Delta_{\alpha \cup \beta} [\lor']
\]

It is fairly obvious that this rule has Distribution built into it. Of course, other rules must be suitably modified. It is easiest to interpret the formulas on a line as grouped into a set so as not to have to worry about ‘structural rules’ corresponding to the commutation and idempotence of conjunction.

The rules →I, →E, ∨I, ∨E, ∧I, and ∧E can all be left as they were (or except for →I and →E, trivially generalised so as to allow for the fact that the premises might be occurring on a line with several other ‘irrelevant’ premises), but we do need one new structural rule:

\[
\Gamma_\alpha \\
\Delta_\alpha [\text{Comma } I]
\]

Once we have this it is natural to take the conjunction rules in ‘Ketonen form’:

\[
\Gamma, A, B_\alpha [\land I'] \\
\Gamma, A \land B_\alpha [\land I']
\]

\[
\Gamma, A \land B_\alpha [\land E'] \\
\Gamma, A, B_\alpha [\land E']
\]

with the rule

\[
\Gamma, \Delta_\alpha [\text{Comma } E]
\]
It is merely a tedious exercise for the reader to show that this new system \(N'\mathbf{R}\) is equivalent to \(N\mathbf{R}\). Incidentally, \(N'\mathbf{R}\) was suggested by reflection upon the Gentzen System \(L\mathbf{R}^+\) of Section 4.9.

Before leaving the question of natural deduction for \(\mathbf{R}\), we would like to mention one or two technical aspects. First, the system of Prawitz [1965] differs from \(\mathbf{R}\) in that it lacks the rule of Distribution. This is perhaps compensated for by the fact that Prawitz can prove a normal form theorem for proofs in his system. A different system yet is that of [Pottinger, 1979], based on the idea that the correct \(\land I\) rule is

\[
\frac{A_\alpha \quad B_\beta}{A \land B_{\alpha\beta}}
\]

He too gets a normal form theorem. We conjecture that some appropriate normal form theorem is provable for the system \(N'\mathbf{R}^+\) on the well-known analogy between cut-elimination and normalisation and the fact that cut-elimination has been proven for \(L\mathbf{R}^+\) (cf. Section 4.9). Negation though would seem to bring extra problems, as it does when one is trying to add it to \(L\mathbf{R}^+\).

One last set of remarks, and we close the discussion of natural deduction. The system \(N\mathbf{R}\) above differs from the natural deduction system for \(\mathbf{R}\) of Anderson and Belnap [1975]. Their system is a so-called ‘Fitch-style’ formalism, and so named \(F\mathbf{R}\). The reader is presumed to know that in this formalism when a hypothesis is introduced it is thought of as starting a subproof, and a line is drawn along the left of the subproof (or a box is drawn around the subproof, or some such thing) to demarcate the scope of the hypothesis. If one is doing a natural deduction system for classical or intuitionistic logic, subproofs or dependency numerals can either be used to do essentially the same job of keeping track the use of hypotheses (though dependency numerals keep more careful track, and that is why they are so useful for relevant implication).

Mathematically, a Fitch-style proof is a nested structure, representing the fact that subproofs can contain further subproofs, etc. But once one has dependency numerals, this extra structure, at least for \(\mathbf{R}\), seems otiose, and so we have dispensed with it. The story for \(\mathbf{E}\) is more complex, since on the Anderson and Belnap approach \(\mathbf{E}\) differs from \(\mathbf{R}\) only in what is allowed to be ‘reiterable’ into subproof. Since implication in \(\mathbf{E}\) is necessary as well as relevant, the story is that in deducing \(B\) from \(A\) in order to show \(A \rightarrow B\), one should only be allowed to use items that have been assumed to be necessarily true, and that these can be taken to be formulas of the form \(C \rightarrow D\). So only formulas of this form can be reiterated for use in the subproof from \(A\) to \(B\). Working out how best to articulate this idea using
only dependency numerals (no lines, boxes, etc.) is a little messy. This concern to keep track of how premises are used in a proof by way of labels has been taken up in a general way by recent work on *Labelled Deductive Systems* [D’Agostino and Gabbay, 1994, Gabbay, 1997].

We would be remiss not to mention other formulations of natural deduction systems for relevance logics and their cousins. A different generalisation of Hunter’s natural deduction systems (which follows more closely the Gentzen systems for positive logics — see Section 4.9) is in [Read, 1988, Slaney, 1990].\(^*\)

### 1.6 Basic Formal Properties of Relevance Logic

This section contains a few relatively simple properties of relevance logics, proofs for which can be found in [Anderson and Belnap, 1975]. With one exception (the ‘Ackermann Properties’ — see below), these properties all hold for both the system R and E, and indeed for most of the relevance logics defined in Section 1.3. For simplicity, we shall state these properties for sentential logics, but appropriate versions hold as well for their first-order counterparts.

First we examine the **Replacement Theorem** *For both R and E,*

\[ \vdash (A \iff B) \land t \rightarrow (\chi(A) \iff \chi(B)). \]

Here \(\chi(A)\) is any formula with perhaps some occurrences of \(A\) and \(\chi(B)\) is the result of perhaps replacing one or more of those occurrences by \(B\). The proof is by a straightforward induction on the complexity of \(\chi(A)\), and one clear role of the conjoined \(t\) is to imply \(\chi \rightarrow \chi\) when \(\chi(= \chi(A))\) contains no occurrences of \(A\), or does but none of them is replaced by \(B\). It might be thought that if these degenerate cases are ruled out by requiring that some actual occurrence of \(A\) be replaced by \(B\), then the need for \(t\) would vanish. This is indeed true for the implication-negation (and of course the pure implication) fragments of \(R\) and \(E\), but not for the whole systems in virtue of the non-theoremhood of what \(V\). Routley has dubbed ‘Factor’:

1. \((A \rightarrow B) \rightarrow (A \land \chi \rightarrow B \land \chi)\).

Here the closest one can come is to

2. \((A \rightarrow B) \land t \rightarrow (A \land \chi \rightarrow B \land \chi)\),

\(^*\)The reader should be informed that still other natural deduction formalisms for \(R\) of various virtues can be found in [Meyer, 1979b] and [Meyer and McRobbie, 1979].
the conjoined $g$ giving the force of having $\chi \rightarrow \chi$ in the antecedent, and the theorem $(A \rightarrow B) \land (\chi \rightarrow \chi) \rightarrow (A \land \chi \rightarrow B \land \chi)$ getting us home. (2) of course is just a special case of the Replacement Theorem. Of more ‘relevant’ interest is the

**Variable Sharing Property.** If $A \rightarrow B$ is a theorem of $R$ (or $E$), then there exists some sentential variable $p$ that occurs in both $A$ and $B$. This is understood by Anderson and Belnap as requiring some commonality of meaning between antecedent and consequent of logically true relevant implications. The proof uses an ingenious logical matrix, having eight values, for which see [Anderson and Belnap, 1975, Section 22.1.3]. There are discussed both the original proof of Belnap and an independent proof of Dončenko, and strengthening by Maksimova. Of modal interest is the

**Ackermann Property.** No formula of the form $A \rightarrow (B \rightarrow C)$ ($A$ containing no $\rightarrow$) is a theorem of $E$. The proof again uses an ingenious matrix (due to Ackermann) and has been strengthened by Maksimova (see [Anderson and Belnap, 1975, Section 22.1.1 and Section 22.1.2]) (contributed by J. A. Coffa) on ‘fallacies of modality’.

### 1.7 First-degree Entailments

A zero degree formula contains only the connectives $\land, \lor, \neg$, and can be regarded as either a formula of relevance logic or of classical logic, as one pleases. A first degree implication is a formula of the form $A \rightarrow B$, where both $A$ and $B$ are zero-degree formulas: Thus first degree implications can be regarded as either a restricted fragment of some relevance logic (say $R$ or $E$) or else as expressing some metalinguistic logical relation between two classical formulas $A$ and $B$. This last is worth mention, since then even a classical logician of Quinean tendencies (who remains unconverted by the considerations of Section 1.2 in favour of nested implications) can still take first degree logical relevant implications to be legitimate.

A natural question is what is the relationship between the provable first-degree implications of $R$ and those of $E$. It is well-known that the corresponding relationship between classical logic and some normal modal logic, say $S4$ (with the $\rightarrow$ being the material conditional and strict implication, respectively), is that they are identical in their first degree fragments. The same holds of $R$ and $E$ (cf. [Anderson and Belnap, 1975, Section 2.42]).

This fragment, which we shall call $R_{\text{fde}}$ (Anderson and Belnap [1975] call it $E_{\text{fde}}$) is stable (cf. [Anderson and Belnap, 1975, Section 7.1]) in the sense that it can be described from a variety of perspectives. For some semantical perspectives see Sections 3.3 and 3.4. We now consider some syntactical perspectives of more than mere ‘orthographic’ significance.
The perhaps least interesting of these perspectives is a ‘Hilbert-style’ presentation of $\mathbf{R}_{\text{fde}}$ (cf. [Anderson and Belnap, 1975, Section 15.2]). It has the following axioms:

3. $A \land B \rightarrow A, A \land B \rightarrow B$  
   Conjunction Elimination
4. $A \rightarrow A \lor B, B \rightarrow A \lor B$  
   Disjunction Introduction
5. $A \land (B \lor C) \rightarrow (A \land B) \lor C$  
   Distribution
6. $A \rightarrow \neg\neg A, \neg\neg A \rightarrow A$  
   Double Negation

It also has gobs of rules:

7. $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$  
   Transitivity
8. $A \rightarrow B, A \rightarrow C \vdash A \rightarrow B \land C$  
   Conjunction Introduction
9. $A \rightarrow C, B \rightarrow C \vdash A \lor B \rightarrow C$  
   Disjunction Introduction
10. $A \rightarrow B \vdash \neg B \rightarrow \neg A$  
    Contraposition.

More interesting is the characterisation of Anderson and Belnap [1962b, 1975] of $\mathbf{R}_{\text{fde}}$ as ‘tautological entailments’. The root idea is to consider first the ‘primitive entailments’.

11. $A_1 \land \ldots \land A_m \rightarrow B_1 \lor \ldots \lor B_n,$
    where each $A_i$ and $B_j$ is either a sentential variable or its negate (an ‘atom’) 
    and make it a necessary and sufficient criterion for such a primitive entail- 
    ment to hold that same $A_i$ actually be identically the same formula as some $B_j$ (that the entailment be ‘tautological’ in the sense that $A_i$ is repeated). 
    This rules out both

12. $p \wedge \neg p \rightarrow q,$
13. $p \rightarrow q \lor \neg q,$

where there is no variable sharing, but also such things as

14. $p \wedge \neg p \land q \rightarrow \neg q,$

where there is (of course all of (12)–(14) are valid classically, where a primit- 
    ive entailment may hold because of atom sharing or because either the 
    antecedent is contradictory or else the consequent is a logical truth).

Now the question remains as to which non-primitive entailments to count as valid. Both relevance logic and classical logic agree on the standard count as valid. Both relevance logic and classical logic agree on the stan-
    dard ‘normal form equivalences’: commutation, association, idempotence,
distribution, double negation, and de Morgan’s laws. So the idea is, given a candidate entailment \( A \rightarrow B \), by way of these equivalences, \( A \) can be put into disjunctive normal form and \( B \) may be put into conjunctive normal form, reducing the problem to the question of whether the following is a valid entailment:

15. \( A_1 \lor \cdots \lor A_k \rightarrow B_1 \land \cdots \land B_h \).

But simple considerations (on which both classical and relevance logic agree) having to do with conjunction and disjunction introduction and elimination show that (15) holds if for each disjunct \( A_i \) and conjunct \( B_j \), the primitive entailment \( A_i \rightarrow B_j \) is valid. For relevance logic this means that there must be atom sharing between the conjunction \( A_i \) and the disjunction \( B_j \).

This criterion obviously counts the Disjunctive Syllogism

16. \( \neg p \land (p \lor q) \rightarrow q \),

as an invalid entailment, for using distribution to put its antecedent into disjunctive normal form, (16) is reduced to

16’ \( (\neg p \land p) \lor (\neg p \land q) \rightarrow q \).

But by the criterion of tautological entailments,

17. \( \neg p \land p \rightarrow q \),

which is required for the validity of (16’), is rejected.

Another pleasant characterisation of \( \mathbf{R}_{\text{ fis }} \) is contained in [Dunn, 1976a] using a simplification of Jeffrey’s ‘coupled trees’ method for testing classically valid entailments. The idea is that to test \( A \rightarrow B \) one works out a truth-tree for \( A \) and a truth tree for \( B \). One then requires that every branch in the tree for \( A \) ‘covers’ some branch in the tree for \( B \) in the sense that every atom in the covered branch occurs in the covering branch. This has the intuitive sense that every way in which \( A \) might be true is also a way in which \( B \) would be true, whether these ways are logically possible or not, since ‘closed’ branches (those containing contradictions) are not exempt as they are in Jeffrey’s method for classical logic. This coupled-trees approach is ultimately related to the Anderson–Belnap tautological entailment method, as is also the method of [Dunn, 1980b] which explicates an earlier attempt of Levy to characterise entailment (cf. also [Clark, 1980]).

1.8 Relations to Familiar Logics

There is a sense in which relevance logic contains classical logic.
ZDF Theorem (Anderson and Belnap [1959a]). The zero-degree formulas (those containing only the connectives $\land, \lor, \neg$) provable in $R$ (or $E$) are precisely the theorems of classical logic.

The proof went by considering a ‘cut-free’ formulation of classical logic whose axioms are essentially just excluded middles (which are theorems of $R / E$) and whose rules are all provable first-degree relevant entailments (cf. Section 2.7). This result extends to a first-order version [Anderson and Belnap Jr., 1959b]. (The admissibility of $\gamma$ (cf. Section 2) provides another route to the proof to the ZDF Theorem.)

There is however another sense in which relevance logic does not contain classical logic:

FACT (Anderson and Belnap [1975, Section 25.1]). $R$ (and $E$) lack as a derivable rule Disjunctive Syllogism:

$$\neg A, A \lor B \vdash B.$$ 

This is to say there is no deduction (in the standard sense of Section 1.4) of $B$ from $\neg A$ and $A \lor B$ as premises. This is of course the most notorious feature of relevance logic, and the whole of Section 2 is devoted to its discussion.

Looking now in another direction, Anderson and Belnap [1961] began the investigation of how to translate intuitionistic and strict implication into $R$ and $E$, respectively, as ‘enthymematic’ implication. Anderson and Belnap’s work presupposed the addition of propositional quantifies to, let us say $R$, with the subsequent definition of ‘$A$ intuitionistically implies $B$’ (in symbols $A \triangleright B$) as $\exists p (p \land (A \land p \rightarrow B))$. This has the sense that $A$ together with some truth relevantly implies $B$, and does seem to be at least in the neighbourhood of capturing Heyting’s idea that $A \triangleright B$ should hold if there exists some ‘construction’ (the $p$) which adjoined to $A$ ‘yields’ (relevant implication) $B$. Meyer in a series of papers [1970a, 1973] has extended and simplified these ideas, using the propositional constant $t$ in place of propositional quantification, defining $A \triangleright B$ as $A \land t \rightarrow B$. If a propositional constant $F$ for the intuitionistic absurdity is introduced, then intuitionistic negation can be defined in the style of Johansson as $\neg A =_{df} A \triangleright F$. As Meyer has discovered one must be careful what axiom one chooses to govern $F$. $F \rightarrow A$ or even $F \triangleright A$ is too strong. In intuitionistic logic, the absurd proposition intuitionistically implies only the intuitionistic formulas, so the correct axiom is $F \triangleright A^*$, where $A^*$ is a translation into $R$ of an intuitionistic formula. Similar translations carry $S4$ into $E$ and classical logic into $R$. 
