Normality, Non-Contamination and Logical Depth in Classical Natural Deduction Technical Report, Part I

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Abstract

In this two-part paper we provide a detailed proof-theoretical analysis of a natural deduction system for classical propositional logic that (i) represents classical proofs in a more natural way than standard Gentzen-style classical natural deduction, (ii) admits of a simple normalization procedure such that normal proofs enjoy the Weak Subformula Property, (iii) provides the means to prove a Non-Contamination Property of normal proofs that is not satisfied by normal proofs in the Gentzen tradition and is useful for applications, especially for formal argumentation, (iv) naturally leads to defining a notion of *depth* of a proof, to the effect that, for every fixed natural k, normal k-depth deducibility is a tractable problem and converges to classical deducibility as k tends to infinity.

1 Introduction

Gentzen introduced his natural deduction systems NJ and NK in his [1935] paper in order to "set up a formal system that comes as close as possible to actual reasoning" [p. 68]. In the same paper, Gentzen also introduced his sequent calculi LJ and LK to overcome some technical difficulties in the proof of his "main result" (*Haupsatz*), namely the subformula theorem for first order logic. The theorem ensures that the search for proofs can be pursued by *analytic* methods, i.e. by considering only inference steps involving formulae that are "contained" in the assumptions or in the conclusion.[1] Thus, no particular ingenuity is required, at least in principle, to construct such analytic arguments and their search is amenable to algorithmic treatment.

[1]"No concepts enter into the proof other than those contained in its final result, and their use was therefore essential to the achievement of that result" [Szabo, 1969, p. 69]; "the final result is, as it were, gradually built up from its constituent elements" [Szabo, 1969, p. 88].

Gentzen proved the subformula property in the context of his sequent calculi by means of the celebrated "cut-elimination" theorem. The proof of the same result for the natural deduction systems, usually ascribed to Dag Prawitz [1965],[2] makes use of a "normalization" procedure by means of which any proof can be reduced to one with a specified (albeit not unique) normal form. Gentzen-style natural deduction is now widely used in the teaching of logic, and has been thoroughly investigated in the philosophical and proof-theoretical literature.[3] In automated deduction circles, however, it has been considerably neglected in favour of alternative methods such as resolution, the sequent calculus, the method of analytic tableaux or variants thereof. The official motivation — namely that natural deduction is less amenable to the development of automated proof search methods — can be, and has been, challenged, especially in view of the normalization theorem and the related subformula property of proofs (see, for example, [Tennant, 1992, Sieg and Byrnes, 1998, Indrzejczak, 2010, Ferrari and Fiorentini, 2015, Maretić, 2018]). Moreover, owing to recent advances in Artificial Intelligence and Human-Oriented Computing, most notably the research program on Formal Argumentation Theory pioneered by [Dung, 1995], there is a growing need for automated proof procedures based on natural inference patterns that, in the spirit of Gentzen's work, come "as close as possible to actual reasoning" and can therefore be fruitfully employed in human-computer interaction [Modgil et al., 2013].

Our starting point in this paper is the well-known fact that Gentzen-style natural deduction is really natural from the point of view of intuitionistic logic, but not from the point of view of classical logic, for its introduction and elimination rules are faithful to the intuitionistic meaning of the logical operators, but not to their classical meaning. As a result they are unable to adequately represent the natural inference patterns that exploit the inner symmetries of classical logic. This is a serious hindrance towards providing adequate alternatives to the more popular methods used in the field of automated deduction for classical logic.

The main purpose of this paper is an in-depth proof-theoretical analysis of a non-standard natural deduction system for classical propositional logic that we call *C-intelim*, which is based on the *classical* truth-table meaning of the logical operators and, while sharing some interesting features with Gentzen-style natural deduction, brings it somewhat closer to the method of analytic tableaux. This allows us to prove a normalization theorem that is more informative than the ones that are usually shown for Gentzen-style systems. The rules of this system were first proposed in [Mondadori, 1989] and discussed in [D'Agostino, 2005].[4] This is, however, the first time these rules are the object of a detailed proof-theoretical investigation aimed at a direct comparison with the large body of literature on Gentzen-style natural deduction.

The restriction of C-intelim to normal proofs enforces a stricter control discipline on proof-construction, to the effect that normal proofs, besides enjoying the *subformula*

[2]In the same year Andrés Raggio also published a short paper containing a proof of Gentzen's *Hauptsatz* [Raggio, 1965]. Ian Von Plato has recently discovered that, in fact, Gentzen had eventually obtained a proof also for his natural deduction systems that he did not publish [von Plato, 2008].

[3] For insightful and comprehensive treatments see [Tennant, 1990, Negri and von Plato, 2001]). For the history of natural deduction see [Pelletier and Hazen, 2012].

[4]They are also used in [D'Agostino and Floridi, 2009, D'Agostino et al., 2013, D'Agostino, 2015] in order to characterize various sequences of tractable approximations to classical propositional logic.

property, enjoy also a kind of weak relevance property that we call *non-contamination*. We say that a proof of A depending on Γ is "contaminated" if one of the following conditions holds: (i) A is equal to the special "falsum" symbol $\curlywedge$ (i.e., the proof is refutation of Γ) and, for some $\Delta \subset \Gamma$, the formulae in Δ are syntactically disjoint[5] from those in $\Gamma \setminus \Delta$; (ii) $A \neq \curlywedge$ and, for some $\Delta \subseteq \Gamma$, the formulae in Δ are syntactically disjoint from those in $(\Gamma \setminus \Delta) \cup \{A\}$.

The classical validity of *ex-falso quodlibet*, or better, *ex-contradictione quodlibet* — from an inconsistent Γ infer any arbitrary conclusion A — implies that for any natural deduction system that is complete for classical logic there are inferences that admit only of contaminated proofs. However, in *normal* C-intelim this can happen only in the trivial case in which the proof is *essentially a (non-contaminated) refutation* of Γ, that is, immediately obtained from such a refutation by means of a peculiar use of the *ex-falso quodlibet* principle. The paradigmatic example is:

$$\frac{\dfrac{A \quad \neg A}{\curlywedge}}{B} \tag{1}$$

where the conclusion is obtained from a non-contaminated refutation of the same assumptions by means of an application of *ex-falso* that is meaningless from the point of view of deductive practice.[6] If this is the case, we call such a proof *improper* and show that it can always be readily transformed into a normal non-contaminated proof of $\curlywedge$ depending on the *same* assumptions, by simply removing the bizarre applications of *ex-falso*. On the other hand, *proper* normal proofs are always non-contaminated and so satisfy the requirement, often called the *variable-sharing* property, that their premises are never syntactically disjoint from their conclusion (except for the special case in which the conclusion is $\curlywedge$ and the proof is, therefore, a non-contaminated refutation of the assumptions).

Thus, our normalization theorem allows us to show that the restriction of C-intelim to normal proofs enjoys the following

> *Non-contamination property*: if π is a proof of A depending on assumptions Γ, then either π is non-contaminated or π is improper. (NCP)

In the latter case there is a straightforward (linear time) procedure to turn π into a non-contaminated proof of $\curlywedge$ depending on the *same* set Γ of assumptions. To the best of our knowledge, NCP and a notion of normal proof that automatically satisfies it are investigated in this paper for the first time.

NCP is not enforced by the standard notion(s) of normal proof in Gentzen-style natural deduction for classical logic. We maintain that a control discipline on the (automated) generation of proofs that enforces this property is of considerable potential interest for a variety of application areas in that it stops the generation of obviously redundant proofs in which a subset of the assumptions on which the conclusion depends

[5]Two formulae are syntactically disjoint when they share no atomic subformula.

[6]As Michael Dummett once put it: "Obviously, once a contradiction has been discovered, no one is going to go *through it*: to exploit it to show that the train leaves at 11:52 or that the next Pope will be a woman." [Dummett, 1991, p. 209].

are totally unrelated to the other assumptions and to the conclusion. (This point will be discussed in Section 2 of Part II.) A prominent research area in which NCP is of crucial importance is, again, that of Formal Argumentation Theory that is now widely regarded as a most promising research program in Artificial Intelligence [Bench-Capon and Dunne, 2007].

Moreover, our normalization theorem leads to a straightforward definition of a measure of *depth* for *normal* natural deduction proofs in such a way that, for every fixed k, if a proof of depth $\leq k$ exists, it can be found in polynomial time. The resulting notion of depth-bounded natural deduction lays the foundations for modelling the deductive practice of resource-bounded agents that reason according to classical logic. We maintain that this approach looks promising in view of practical applications of Argumentation Theory to real-world, resource-bounded agents [D'Agostino and Modgil, 2016, 2017]. From this point of view, this paper is a further articulation of a research program whose philosophical and computational aspects have been investigated in [D'Agostino and Floridi, 2009, D'Agostino et al., 2013, D'Agostino, 2014, 2015]. Here, the focus is on the proof-theoretical presentation, on the normalization theorem — with three different"shades" of normal proofs each of which is interesting in its own right — and on the NCP of normal proofs.

Part I is organized as follows. In Section 2 we discuss the drawbacks of Gentzen-style natural deduction from the point of view of classical logic. In Section 3 we present the C-intelim system and discuss its main features. The normalization theorem is proved in two steps. In Section 4 we discuss the intermediate notion of *quasi-normal* proof and argue that it has a proof-theoretical interest on its own, in that it allows representations of proofs that exclude trivial detours, while not necessarily being "analytic", and that may, in some cases, be significantly shorter than any analytic proof.

Part II is organized as follows. In Section 1 we define the notion of *normal* proof and show that it enjoys the weak subformula property (every formula occurring in a normal proof of A depending on a set Γ of assumptions is either a subformula of some formula in $\Gamma \cup \{A\}$, or the negation of such a subformula, or is equal to $\curlywedge$). Next, in Sections 2 and 3 we discuss the contamination problem and introduce the distinction between *proper* and *improper* normal proofs, showing that proper normal proofs enjoy the variable-sharing property, are refutation-complete and are also complete with respect to all classical consequences of consistent sets of assumptions. In Section 4 we discuss the correspondence between the C-intelim system analysed in this paper and the tableau-like method used in [D'Agostino, 2015] that here we call *C-intelim Tableaux*. We argue that, in the context of classical logic, the latter could provide a smooth transition to implementable algorithms for the search of natural deduction proofs that may successfully compete with the more popular alternatives based on resolution, sequent calculi or tableaux. This further supports the promise of this approach for use by real-world, resource-bounded agents. (See also the concluding section of Part II on this point.) Finally, in Section 5 we report some complexity facts about C-intelim, and in particular that the notion of normal k-depth C-intelim deducibility provides a hierarchy of tractable approximation to classical propositional logic.

In this paper we restrict ourselves to propositional logic. Extending our main results to first-order logic would be relatively unproblematic if we attached no importance to the notion of depth-bounded reasoning. In the first-order case such a notion requires incorporating some of the ideas put forward by Jaakko Hintikka (e.g., in [Hintikka, 1972]) to provide an analogous hierarchy of depth-bounded approximations to first-oder logic. Since a characterization of depth-bounded reasoning is crucial in view of practical applications in a variety of areas we shall postpone these investigations until a subsequent paper. Throughout this paper we assume that the reader is familiar with the basic notions of Gentzen-style natural deduction.[7]

2 Is Gentzen-style natural deduction really natural for classical logic?

It is well-known that Gentzen-style natural deduction provides a natural formalization of intuitionistic logic, but a quite unnatural formalization of classical logic. Gentzen himself observed that his classical calculus *NK* was obtained by adding to the intuitionistic calculus *NJ* the law of excluded middle in "a purely external manner" that spoiled the harmony between introductions and eliminations [Szabo, 1969, p. 81]. This approach was illuminating in that it clarified the relationship between the two logical systems, but did so from the vantage point of intuitionistic logic. It is, therefore, only to be expected that the *NK* proof of an inference that is classically, but not intuitionistically, valid turns out to be rather unnatural The same holds true for Prawitz's variant [Prawitz, 1965], that consists in replacing the intuitionistic *ex-falso* rule and the law of excluded middle with *classical reductio*. Prawitz's rules for natural deduction are shown in Table 1.[8] Recall that in some of the rules, the vertical dots stand for a natural deduction proof of the formula below the dots depending on assumptions that may include the ones enclosed in square brackets. The latter are "discharged" by the application of the rule in the sense that the conclusion no longer *depends* on them, but only on the yet undischarged assumptions that occur in the leaves. We call *proof rules* those rules that involve the discharge of assumptions and *inference rules* those rules that do not. So, in this system, the inference rules are $\wedge\mathcal{I}$, $\wedge\mathcal{E}$, $\vee\mathcal{I}$, $\to\mathcal{E}$, $\neg\mathcal{E}$ and $\curlywedge_{\mathsf{I}}$, while the proof rules are $\vee\mathcal{E}$, $\to\mathcal{I}$, $\neg\mathcal{I}$. It is important to note that in the proof rules assumptions may be discharged *vacuously*, i.e., even if they do not occur in the leaves of the proof tree. A system for classical logic is obtained by replacing the inference rule $\curlywedge_{\mathsf{I}}$ with the proof rule $\curlywedge_{\mathsf{C}}$.[9] Note that the intuitionistic falsum rule $\curlywedge_{\mathsf{I}}$ is a special case of the classical one that arises when the assumption $\neg A$ is vacuously discharged.

A proof of A *depending* on Γ is a tree of occurrences of formulae constructed in accordance with the rules[10] such that A occurs at the root and Γ is the set of all the undischarged assumptions occurring at the leaves. The system is complete for classical logic in the sense that for every Δ, A such that A is a classical consequence of Δ, then there is a natural deduction proof of A depending on some $\Gamma \subseteq \Delta$.

[7] For references see footnote 3 above.

[8] For an explanation of the rules the reader can refer to Prawitz's book or to other expositions, for example those cited in footnote 3.

[9] As mentioned above, Gentzen's system *NK* used the law of excluded middle $A \vee \neg A$ as an axiom, i.e. an extra assumption that can be freely introduced in a proof, instead of $\curlywedge_{\mathsf{C}}$. He also remarked that this axiom could have been equivalently replaced by the inference rule $\neg\neg A/A$.

[10] For a formal definition see [Prawitz, 1965].

INTRODUCTION RULES

$$\frac{A \quad B}{A \wedge B}\wedge\mathcal{I} \qquad \frac{A}{A \vee B}\vee\mathcal{I}1 \qquad \frac{B}{A \vee B}\vee\mathcal{I}2$$

$$\frac{\begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \to B}\to \mathcal{I} \qquad \frac{\begin{array}{c}[A]\\ \vdots \\ \curlywedge\end{array}}{\neg A}\neg\mathcal{I}$$

ELIMINATION RULES

$$\frac{A \wedge B}{A}\wedge\mathcal{E}1 \qquad \frac{A \wedge B}{B}\wedge\mathcal{E}2 \qquad \frac{A \vee B \quad \begin{array}{c}[A]\\ \vdots \\ C\end{array} \quad \begin{array}{c}[B]\\ \vdots \\ C\end{array}}{C}\vee\mathcal{E}$$

$$\frac{A \to B \quad A}{B}\to\mathcal{E} \qquad \frac{\neg A \quad A}{\curlywedge}\neg\mathcal{E}$$

FALSUM RULES

$$\frac{\curlywedge}{A}\curlywedge_{\mathsf{I}} \qquad \frac{\begin{array}{c}[\neg A]\\ \vdots \\ \curlywedge\end{array}}{A}\curlywedge_{\mathsf{C}}$$

Table 1: Prawitz's rules for natural deduction.

Let us consider a proof in Prawitz's system of the intuitionistically valid inference:

$$\neg A \vee \neg B \vdash \neg(A \wedge B) \tag{2}$$

We use numerals to keep track of the assumptions that are discharged by the application of a proof rule. The numerals corresponding to the discharged assumptions are shown beside the inference line.

$$\dfrac{\neg A \vee \neg B \qquad \dfrac{\dfrac{[\neg A]^1 \qquad \dfrac{[A \wedge B]^3}{A}}{\curlywedge}}{\neg(A \wedge B)}\,{}^{3} \qquad \dfrac{\dfrac{[\neg B]^2 \qquad \dfrac{[A \wedge B]^3}{B}}{\curlywedge}}{\neg(A \wedge B)}\,{}^{3}}{\neg(A \wedge B)}\,{}^{1,2} \tag{3}$$

Let us now consider a standard proof of the reverse inference:

$$\neg(A \wedge B) \vdash \neg A \vee \neg B \tag{4}$$

which is *not* intuitionistically valid.

$$\dfrac{\dfrac{\neg(A \wedge B) \qquad \dfrac{\dfrac{\dfrac{[\neg(\neg A \vee \neg B)]^1 \qquad \dfrac{[\neg A]^2}{\neg A \vee \neg B}}{\curlywedge}}{A}\,{}^{2} \qquad \dfrac{\dfrac{[\neg(\neg A \vee \neg B)]^1 \qquad \dfrac{[\neg B]^3}{\neg A \vee \neg B}}{\curlywedge}}{B}\,{}^{3}}{A \wedge B}}{\curlywedge}}{\neg A \vee \neg B}\,{}^{1} \tag{5}$$

This proof is quite unnatural from the classical point of view in that it does not exploit the inner symmetries of classical logic. Indeed, it is *very different* from the previous one. However, in a classical setting, the two proofs should essentially be the same, modulo the duality of $\vee$ and $\wedge$. By contrast, in both the classical sequent calculus LK and the Tableau method the proofs of (2) and (4) have essentially the same structure.

As these two examples strongly suggest, standard natural deduction is really natural from the point of view of intuitionistic logic, but is not so natural from the point of view of classical logic. If we are interested in a deduction system that is really natural for classical logic, we need introduction and elimination rules that closely reflect the classical meaning of the logical operators and the way in which these are used in classical proofs. In such a system a formula and its negation should be treated symmetrically. Moreover, the conjunction and disjunction operators should be governed by dual rules. This is the case, for instance, with the tableau method, where we have tableau rules for a compound formula of a given logical form and for its negation and where the rules for $\wedge$ and $\vee$ are dual of each other.[11] However, the interplay between introduction and elimination rules as well as the possibility of generating direct proofs, that are typical of natural deduction, are inevitably lost in the tableau method that uses only elimination rules, so that a proof of a conclusion from a set of assumptions is obtained by refuting its negation on the basis of the assumptions.

3 C-intelim deduction

The C-intelim system is a natural deduction system whose rules, unlike those of standard natural deduction, are faithful to the classical meaning of the logical operators (i.e., to their truth-table interpretation) and not to their intuitionistic meaning. It was introduced in [Mondadori, 1989] and further investigated in [D'Agostino, 2005].[12] The C-intelim rules are shown in Table 2.

For the sake of philosophical analysis, the rules are best presented in terms of *signed formulae* of the form $T\,A$ or $F\,A$, with A an ordinary formula. The standard reading of these signed formulae is "A is true" and "A is false". In the context of classical logic this appears as the most natural way of achieving "purity" (each rule deals only with one logical operator) and "separation" (a proof should make use only of intelim rules for the operators that occur in the premises or in the conclusion), as well as the stronger form of separation that is embodied in the subformula property.[13] (For a well-argued philosophical defence of the use of signed formulae in the proof theory of classical logic the reader is referred to [Bendall, 1978].) This is also the approach followed by Smullyan [1968] in his presentation of the tableau method. More recently, the use of signed formulae for philosophical purposes has been central in the discussion on "bilateralism" [Smiley, 1996, Rumfit, 2000, Humberstone, 2000, Ferreira, 2008, Gabbay, 2017] as an inferential approach to the meaning of the classical operators. On the other hand, without essential extensions of the logical language (signed formulae) or of the intuitive notion of inference rule (multi-conclusion sequent calculus), most authors are skeptical about the possibility of a genuine inferential semantics for classical logic.[14]

Note that in C-intelim the introduction and elimination rules for the logical operators, as well as the falsum rules, are all *inference rules*, involving no discharge of assumptions. The only *proof rule* RB is a structural rule that expresses the classical principle of bivalence and is therefore peculiar to classical logic. For each application

[11] Smullyan once claimed that tableaux could indeed be presented as a sort of natural deduction for classical logic ([Smullyan, 1965]). Dual rules are also used in the natural deduction system EN^* proposed by Kent Bendall in [1978], who also advocates the use of signed formulae to solve the separation problem for classical logic. (On this point see also Section 3 below.)

[12] In [D'Agostino and Floridi, 2009, D'Agostino et al., 2013, D'Agostino, 2015] similar rules are used in a different format to define a hierarchy of tractable depth-bounded approximations to classical propositional logic.

[13] An alternative, but closely related, way of achieving the same results in terms of ordinary formulae is that of resorting to *multi-conclusion sequents* as the primary components of a deduction system. The connection between multi-conclusion sequents and signed formulae is made apparent when looking at the translation of classical sequent proofs in systems with no structural rules like Kleene's G4 [Kleene, 1967, chapter 6] into closed semantic tableaux. See [D'Agostino, 1990, Section 2.2] for the details.

[14] One notable exception is [Sandqvist, 2009]; see also the analysis in [Makinson, 2013].

of RB the formula A that occurs in the discharged assumptions is called the *RB-formula* of that application. A C-intelim proof of A *depending* on Γ is a tree of occurrences of signed formulae constructed in accordance with the C-intelim rules such that $T\,A$ occurs at the root and $\{T\,B \mid B \in \Gamma\}$ is the set of all undischarged assumptions that occur at the leaves.

Observe that the intelim rules for disjunction and conjunction are dual of each other, and that a signed formula and its conjugate are treated symmetrically, as they should be in a classical setting. For each logical operator, we have intelim rules for a signed formula containing it as main operator as well as intelim rules for its conjugate. This feature is shared by the tableau method and other bilateral systems of deduction, such as Bendall's [Bendall, 1978] or Rumfit's [Rumfit, 2000]. Unlike the tableau method, however, C-intelim contains introduction as well as elimination rules and so can be used for direct proofs as well as for refutations. Moreover, the elimination rules for $T\,A \to B$, $T\,A \vee B$ and $F\,A \wedge B$ do not generate any branching and are standard inference rules that require an additional minor premise. So, unlike Gentzen-style natural deduction and other "bilateral" natural deduction systems, C-intelim contains no discharge rule for the logical operators.

The rules RNC (Rule of Non-Contradiction) and XFQ (Ex-Falso Quodlibet) are inference rules that express the basic principle of classical semantics according to which no valuation can make the same formula both true and false. We find it convenient to represent this principle by making use of the "falsum" symbol $\curlywedge$. Throughout this paper we make the simplifying assumption that "$\curlywedge$" occurs only in the context of the falsum rules, namely as conclusion or RNC or as premise of XFQ, and nowhere else. From the point of view of classical logic this is a reasonable stipulation and avoids a good deal of tedious details in the proofs and in the definitions. However, there is no loss of generality in making this simplifying assumption. So, for us $\curlywedge$ will be essentially a marker to make it explicit that a contradiction has been reached. Note that each inference rule expresses the fact that every Boolean valuation that satisfies its premise(s) satisfies also its conclusion.[15] Then, RNC and XFQ are classically correct for the simple reason that no Boolean valuation can satisfy their premises. In our view RB, RNC and XFQ are all *structural rules*, in that they reflect the fundamental properties governing the underlying (classical) notions of truth and falsity — governed by the principles of bivalence and non-contradiction — and not the inferential behaviour of some logical operators.[16]

Under an alternative reading of signed formulae, we may interpret them as assertions about the current information state: $T\,A$ would then mean that we possess the information that A is true and $F\,A$ that we possess the information that A is false. In this view all the inference rules are rules that draw straightforward conclusions from information that *we actually possess*. On the other hand the rule RB simulates alternative information states that extend the actual one. Thus, the discharged assumptions in each application of RB are called *virtual assumptions* in that they represent "virtual information", i.e. information that *we do not actually possess*. The notion of proof-tree

[15] That a Boolean valuation v *satisfies* a signed formula is to be intended in the obvious way: v satisfies $T\,A$ when $v(A) = 1$ and $F\,A$ when $v(A) = 0$.

[16] As explained above, we do not treat $\curlywedge$ as a logical operator.

INTRODUCTION RULES

$$\frac{T\,A \quad T\,B}{T\,A \wedge B}\,T\wedge\mathcal{I} \qquad \frac{F\,A}{F\,A \wedge B}\,F\wedge\mathcal{I}1 \qquad \frac{F B}{F A \wedge B}\,F\wedge\mathcal{I}2$$

$$\frac{F\,A \quad F\,B}{F\,A \vee B}\,F\vee\mathcal{I} \qquad \frac{T\,A}{T\,A \vee B}\,T\vee\mathcal{I}1 \qquad \frac{T\,B}{T\,A \vee B}\,T\vee\mathcal{I}2$$

$$\frac{T\,A \quad F\,B}{F\,A \rightarrow B}\,F\rightarrow\mathcal{I} \qquad \frac{F\,A}{T\,A \rightarrow B}\,T\rightarrow\mathcal{I}1 \qquad \frac{T\,B}{T\,A \rightarrow B}\,T\rightarrow\mathcal{I}2$$

$$\frac{T\,A}{F\,\neg A}\,F\neg\mathcal{I} \qquad \frac{F\,A}{T\,\neg A}\,T\neg\mathcal{I}$$

ELIMINATION RULES

$$\frac{T\,A \vee B \quad F\,A}{T\,B}\,T\vee\mathcal{E}1 \qquad \frac{T\,A \vee B \quad F\,B}{T\,A}\,T\vee\mathcal{E}2 \qquad \frac{F\,A \vee B}{F\,A}\,F\vee\mathcal{E}1 \qquad \frac{F\,A \vee B}{F\,B}\,F\vee\mathcal{E}2$$

$$\frac{F\,A \wedge B \quad T\,A}{F\,B}\,F\wedge\mathcal{E}1 \qquad \frac{F\,A \wedge B \quad T\,B}{F\,A}\,F\wedge\mathcal{E}2 \qquad \frac{T\,A \wedge B}{T\,A}\,T\wedge\mathcal{E}1 \qquad \frac{T\,A \wedge B}{T\,B}\,T\wedge\mathcal{E}2$$

$$\frac{T\,A \rightarrow B \quad T\,A}{T\,B}\,T\rightarrow\mathcal{E}1 \qquad \frac{T\,A \rightarrow B \quad F\,B}{F\,A}\,F\rightarrow\mathcal{E}2 \qquad \frac{F\,A \rightarrow B}{T\,A}\,F\rightarrow\mathcal{E}1 \qquad \frac{F\,A \rightarrow B}{F\,B}\,F\rightarrow\mathcal{E}2$$

$$\frac{T\,\neg A}{F\,A}\,T\neg\mathcal{E} \qquad \frac{F\,\neg A}{T\,A}\,F\neg\mathcal{E}$$

FALSUM RULES

$$\frac{T\,A \quad F\,A}{T\,\curlywedge}\ \text{RNC} \qquad \frac{T\,\curlywedge}{S\,A}\ \text{XFQ}^{*}$$

RULE OF BIVALENCE

$$\frac{\begin{array}{cc} [T\,A] & [F\,A] \\ \vdots & \vdots \\ S\,B & S\,B \end{array}}{S\,B}\ \text{RB}^{*}$$

* under any uniform substitution of S with T or F.

Table 2: The C-intelim rules for signed formulae.

INTRODUCTION RULES

$$\frac{A \quad B}{A \wedge B}\, \wedge\mathcal{I} \qquad \frac{\neg A}{\neg(A \wedge B)}\, \neg\wedge\mathcal{I}1 \qquad \frac{\neg B}{\neg(A \wedge B)}\, \neg\wedge\mathcal{I}2$$

$$\frac{\neg A \quad \neg B}{\neg(A \vee B)}\, \neg\vee\mathcal{I} \qquad \frac{A}{A \vee B}\, \vee\mathcal{I}1 \qquad \frac{B}{A \vee B}\, \vee\mathcal{I}2$$

$$\frac{A \quad \neg B}{\neg(A \to B)}\, \neg\!\to\mathcal{I} \qquad \frac{\neg A}{A \to B}\, \to\mathcal{I}1 \qquad \frac{B}{A \to B}\, \to\mathcal{I}2$$

$$\frac{A}{\neg\neg A}\, \neg\neg\mathcal{I}$$

ELIMINATION RULES

$$\frac{A \vee B \quad \neg A}{B}\, \vee\mathcal{E}1 \qquad \frac{A \vee B \quad \neg B}{A}\, \vee\mathcal{E}2 \qquad \frac{\neg(A \vee B)}{\neg A}\, \neg\vee\mathcal{E}1 \qquad \frac{\neg(A \vee B)}{\neg B}\, \neg\vee\mathcal{E}2$$

$$\frac{\neg(A \wedge B) \quad A}{\neg B}\, \neg\wedge\mathcal{E}1 \qquad \frac{\neg(A \wedge B) \quad B}{\neg A}\, \neg\wedge\mathcal{E}2 \qquad \frac{A \wedge B}{A}\, \wedge\mathcal{E}1 \qquad \frac{A \wedge B}{B}\, \wedge\mathcal{E}2$$

$$\frac{A \to B \quad A}{B}\, \to\mathcal{E}1 \qquad \frac{A \to B \quad \neg B}{\neg A}\, \to\mathcal{E}2 \qquad \frac{\neg(A \to B)}{A}\, \neg\!\to\mathcal{E}1 \qquad \frac{\neg(A \to B)}{\neg B}\, \neg\!\to\mathcal{E}2$$

$$\frac{\neg\neg A}{A}\, \neg\neg\mathcal{E}$$

FALSUM RULES

$$\frac{A \quad \neg A}{\curlywedge}\ \text{RNC} \qquad \frac{\curlywedge}{A}\ \text{XFQ}$$

RULE OF BIVALENCE

$$\frac{\begin{array}{cc}[A] & [\neg A] \\ \vdots & \vdots \\ B & B\end{array}}{B}\ \text{RB}$$

Table 3: The C-intelim rules for unsigned formulae.

based on these rules is essentially the same as that of standard natural deduction, except that a proof of A depending on the undischarged assumptions Γ is replaced by a proof of $T\,A$ depending on the undischarged assumptions $\{T\,B \mid B \in \Gamma\}$.

For all practical purposes, however, we may find it convenient to work with *unsigned formulae*, by exploiting the classical meaning of negation. This amounts to simply removing the sign T before a formula and replacing the sign F with the negation operator. The resulting rules for unsigned formulae are displayed in Table 3. In this version, the intelim rules are no longer "pure", the subformula property holds only in a weaker form (see Section 1 of Part II) and the rules RNC and RB no longer appear to be structural, since they do mention the negation operator. On the other hand, we can look at this version of the rules only as a practically convenient "translation" of the rules for signed formulae into an ordinary logical language, and refer to the original version for all philosophical purposes. In particular, we can still regard RNC and RB (as well as XFQ) as structural rules in that their original version for signed formulae does not mention any logical operator.[17] In the sequel we shall present all our results with reference to the rules for unsigned formulae and leave it to the more philosophically-minded readers to translate them into the original version for signed formulae.

In the unsigned version the two-premise elimination rules correspond to time-honoured principles of inference: *modus ponens*, *modus tollens*, *disjunctive syllogism* and its dual.[18] In the elimination rules, the formula containing the logical operator that is to be eliminated is called *major premise* and the other (if any) is called *minor premise*. The less natural rules, from the point of view of ordinary usage, namely the introduction rules for a true conditional and the elimination rules for a false conditional, are related to the Philonian meaning of this operator which is, however, typical of classical logic. The Philonian conditional, also called "material implication", is defined by $A \to B =_{\mathrm{def}} \neg A \vee B$ or, equivalently, by $A \to B =_{\mathrm{def}} \neg(A \wedge \neg B)$. This is the closest we can get to a "real" conditional operator in the Boolean framework. Clearly, such a conditional can be discarded from the language with no significant loss as far as the the structure of arguments is concerned.

As in Gentzen-Prawitz natural deduction, C-intelim proof of A *depending* on Γ is a tree of occurrences of formulae constructed in accordance with the C-intelim rules such that A occurs at the root and Γ is the set of all undischarged assumptions that occur at the leaves. The tree in Fig. 1 shows a C-intelim deduction of G depending on

$$\{A \to \neg B, B \vee C, \neg(C \wedge \neg B), A \vee E, (E \vee F) \to \neg D, \neg G \to D\}.$$

Note that the last step is an occurrence of RB that discharges the temporary assumptions A (which occurs twice among the leaves of the left subtree) and $\neg A$ (which occurs once among the leaves of the right subtree). This example also shows that the standard format in which proofs are represented, while being very perspicuous, is somewhat inefficient, in that it may involve an unnecessary duplication of assumptions and of

[17] In its version for unsigned formulae RB is sometimes called "Classical Dilemma" and can be used, as in [Tennant, 1990], in addition to the standard intuitionistic rules, to obtain another variant of Gentzen's natural deduction for classical logic.

[18] Chrysippus (III century B.C.) listed these rules among the fundamental indemonstrable principles of reasoning (*anapodeiktoi*), except that he intended disjunction in its exclusive sense.

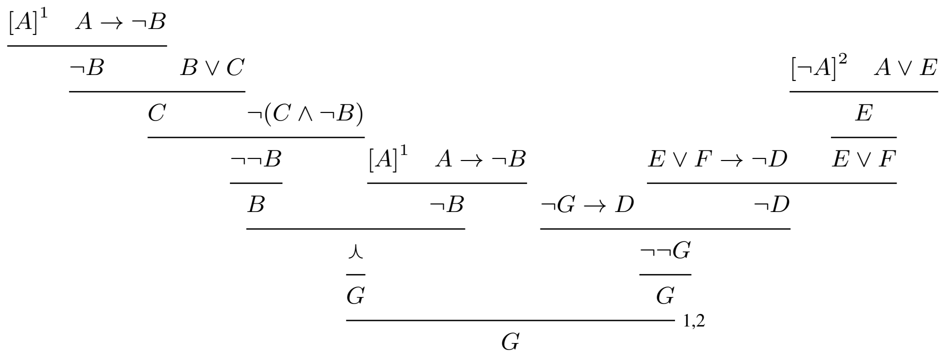

Figure 1: A C-intelim proof.

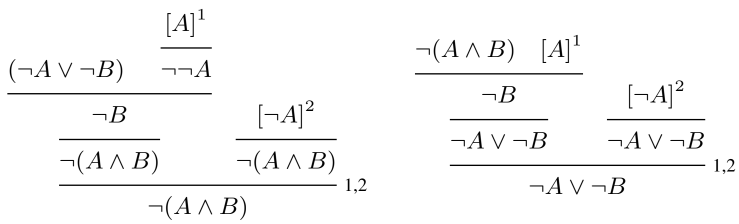

Figure 2: C-intelim proofs of (2) and (4).

identical subproofs. In Part II, Section 4 we shall present a more "streamlined" format for C-intelim proofs that partly avoids the redundancy of the standard format and is more suitable for proof-search algorithms However, the standard format is better suited to proving results on the transformation of proofs. Therefore, se shall stick to in order to provide a clearer presentation of the normalization and non-contamination theorems. The trees in Fig. 2 show C-Intelim proofs of (2) and (4). Note that the proof on the right is much closer to the classical way of reasoning than the standard natural deduction proof shown in (5) and, accordingly, exhibits the same structure as the one on the left.

Definition 3.1. *We say that A is* C-intelim deducible from Γ*, and write* $\Gamma \vdash^{\mathrm{IE}} A$*, if there is a C-intelim proof of A depending on $\Delta \subseteq \Gamma$.*

The completeness of C-intelim for classical propositional logic can be easily shown by simulating the rules of Prawitz's system or the truth-table method (this is left to the reader).[19]

Remark 3.1. *RB can be used, together with the elimination rules, to simulate any of the introduction rules. Conversely, RB can be used together with the introduction rules to simulate any of the elimination rules. To see this it is sufficient to look at the two*

[19]To simulate a truth-table proof of A from Γ it is sufficient to express all possible assignments of truth-values to the atomic formulae occurring in $\Gamma \cup \{A\}$ by means of virtual assumptions introduced via RB, and then use the introduction rules to obtain either $T\,A$, or $F\,B$ for some $B \in \Gamma$. In most cases, it is not necessary to represent all possible assignments to *all* the atomic formulae occurring in $\Gamma \cup \{A\}$.

$$\dfrac{[A\vee B]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg(A\vee B)]^2}{\neg A} \qquad A}{\curlywedge}}{A\vee B}}{A\vee B}\,{}_{1,2} \qquad\qquad \dfrac{[B]^1 \qquad \dfrac{\dfrac{\dfrac{[\neg B]^2 \quad \neg A}{\neg(A\vee B)} \qquad A\vee B}{\curlywedge}}{B}}{B}\,{}_{1,2}$$

Figure 3: Simulating introductions via RB + eliminations and simulating eliminations via RB + introductions.

examples in Figure 3. This clearly implies that RB + Introduction rules and RB + Elimination rules are both complete for classical logic and are essentially equivalent to the systems KI and KE (see Section 4 of Part II for references). However, using both introduction and elimination rules (i) allows for more natural and shorter proofs (although not essentially *shorter because the simulation is clearly polynomial); (ii) it makes a substantial difference when the notion of depth-bounded proof is taken into account, in that it reduces the number of applications of the RB rule (the only discharge rule of the system) that, as we shall see, is key to define the depth of a C-intelim deduction.*

4 Quasi-normal proofs

In this section we introduce the notion of *quasi-normal* proof as a step towards the notion of normal proof that will be introduced in Section 1 of Part 2. However, quasi-normal proofs are of interest in their own right, in that they allow for the representation of *non-analytic proofs* (i.e., proofs that do not enjoy the subformula property) that, however, contain no "detours" and can, in some cases, be much shorter than any analytic proof.[20]

Definition 4.1. *An application of RB is* canonical *in a C-intelim tree $\mathcal{T}$ if there is no application of an inference (intelim or falsum) rule in $\mathcal{T}$ below its conclusion. A proof $\mathcal{T}$ is* RB-canonical *if all applications of RB in it are canonical in $\mathcal{T}$.*

The notion of RB-canonical proof is motivated both by technical reasons concerning the proof of normalization and also by the fact that in RB-canonical proofs there is a clear separation between the components consisting only of applications of inference rules and the final applications of the proof rule RB. Any C-intelim proof can be turned

[20]That analytic proofs may be significantly longer than non-analytic ones is well-known in the literature on the relative complexity of proof systems. Proof systems that generate only analytic proofs, such as the Tableau Method or Resolution, can be polynomially simulated by Frege systems (i.e., standard Hilbert-style axiomatic systems) but cannot polynomially simulate Frege systems. For an overview of these results see [Urquhart, 1995].

into an RB-canonical one by applying the following transformations:[21]

$$\dfrac{\dfrac{\begin{matrix}[A]^1 & [\neg A]^2 \\ \mathcal{T}_1 & \mathcal{T}_2 \\ C & C\end{matrix}}{C}\,{}^{1,2}}{D} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}[A]^1 \\ \mathcal{T}_1 \\ C\end{matrix}}{D} \quad \dfrac{\begin{matrix}[\neg A]^2 \\ \mathcal{T}_2 \\ C\end{matrix}}{D}}{D}\,{}^{1,2} \tag{T1}$$

$$\dfrac{\dfrac{\begin{matrix}[A]^1 & [\neg A]^2 \\ \mathcal{T}_1 & \mathcal{T}_2 \\ C & C\end{matrix}}{C}\,{}^{1,2} \quad \begin{matrix}\mathcal{T}_3 \\ D\end{matrix}}{E} \quad\rightsquigarrow\quad \dfrac{\dfrac{\begin{matrix}[A]^1 & \\ \mathcal{T}_1 & \mathcal{T}_3 \\ C & D\end{matrix}}{E} \quad \dfrac{\begin{matrix}[\neg A]^2 & \\ \mathcal{T}_2 & \mathcal{T}_3 \\ C & D\end{matrix}}{E}}{E}\,{}^{1,2} \tag{T2}$$

The transformations (T1) and (T2) can be applied, respectively, whenever C/D is an instance of a one-premise inference rule, and whenever $C, D/E$ is an instance of a two-premise inference rule. Their repeated application results in pushing downwards all the applications of RB so that, eventually, the conclusion of an application of RB is never used as a premise of an intelim or of a falsum rule and must be identical to the conclusion of the whole proof.[22]

Remark 4.1. *These transformations involve some duplication of occurrences of formulae or of identical subproofs, and therefore an increase in the size of the proof. While in the case of* (T1) *it can be shown that this increase is at most linear in the size of* $\mathcal{T}$*,* (T2) *may lead to exponentially longer proofs.*

Using the transformation (T2) some care must be taken in finding the right parameter which is reduced by each application of the transformations until it drops to the value 0 for which the proof is RB-canonical. Given a subproof $\mathcal{T}'$ of $\mathcal{T}$ ending with an application of RB, let $\mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}')$ be the number of occurrences of formulae below its root that result from an application of an inference (intelim or falsum) rule. Now, we define $d_1(\mathcal{T})$ as follows:

$$d_1(\mathcal{T}) = \sum_{T' \in S} \mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}') \tag{6}$$

where S is the set of *distinct* subproofs of $\mathcal{T}$ ending with an application of RB, that is, the two occurrences of $\mathcal{T}_3$ in (T2) must be counted as one. Observe that each application of the transformations (T1)–(T2) yields a tree $\mathcal{T}'$ such that $d_1(\mathcal{T}') < d_1(\mathcal{T})$. The key observations are: (i) the non RB-canonical subproof ending with C in the lefthand

[21] In the sequel, the transformations must be intended as follows: replace a *subproof* of the form shown on left of "$\rightsquigarrow$" with a subproof of the form shown on its right.

[22] On this reduction to what we call RB-canonical form, see also [Tennant, 1990, p. 94], where it is used to prove that every classically refutable set of assumptions is also intuitionistically refutable (in the propositional case). Tennant uses RB (Classical Dilemma) in addition to the intuitionistic Gentzen rules to characterize classical propositional logic.

side is replaced with the (possibly non RB-canonical) subproof ending with E in the righthand side for which the value of $\mathrm{dnc}_{\mathrm{RB}}$ is strictly less; (ii) all the copies of the possibly non RB-canonical subproofs of $\mathcal{T}_3$ in the righthand side are counted as one. Then, the reader can verify that by repeatedly applying the transformations to a proof of A depending on Γ we eventually obtain an RB-canonical proof of A depending exactly on the same set Γ of assumptions.

Lemma 4.1. *Every C-intelim proof of A depending on Γ can be transformed into an RB-canonical C-intelim proof of A depending on Γ, by means of any sufficiently long sequence of applications of* (T1)–(T2).

Definition 4.2. *Let the* depth *of an RB-canonical C-intelim proof $\mathcal{T}$, denoted by* $\mathsf{depth}(\mathcal{T})$*, be defined as follows:*

1. *If $\mathcal{T}$ contains no application of RB, then* $\mathrm{depth}(\mathcal{T}) = 0$*;*
2. *if T has the form*

$$\frac{\begin{matrix}[A] \\ \mathcal{T}_1 \\ B\end{matrix} \qquad \begin{matrix}[\neg A] \\ \mathcal{T}_2 \\ B\end{matrix}}{B}$$

   then $\mathrm{depth}(\mathcal{T}) = \max(\mathrm{depth}(\mathcal{T}_1), \mathrm{depth}(\mathcal{T}_2)) + 1$.

The notion of depth for RB-canonical proofs emphasizes the role of RB as the only discharge rule that involves the use of "virtual information". Its significance as a proxy for the *intuitive* notion of depth of the reasoning process represented by the proof[23] depends on how the "virtual space", i.e., the set of formulae that may be used as RB-formulae, is bounded. We shall see in Part II that the virtual space can be bounded in a variety of ways and, in particular, can be restricted to the subformulae of the assumptions and of the conclusion of the proof (as required for *normal* proofs).

Definition 4.3. *A* 0-depth component *of an RB-canonical C-intelim proof $\mathcal{T}$ is a maximal* 0*-depth subtree of $\mathcal{T}$, i.e., one that is not a proper subtree of any* 0*-depth subtree of $\mathcal{T}$.*

Note that in an RB-canonical proof $\mathcal{T}$ the conclusion of every 0-depth component is the conclusion of $\mathcal{T}$ itself. The general structure of an RB-canonical k-depth C-intelim proof of A depending on Γ is illustrated in Figure 4. The triangles labelled with $\mathcal{T}_1, \ldots, \mathcal{T}_n$ represent the 0-depth components of the proof. Each 0-depth component $\mathcal{T}_i$ is a proof of A depending on $\Gamma_i \cup \Delta_i$ such that: (i) $\Gamma_i \subseteq \Gamma$, (ii) Δ_i, with $|\Delta_i| \leq k$, is the set of virtual assumptions introduced in $\mathcal{T}_i$ that are subsequently discharged in $\mathcal{T}$ via applications of RB, (iii) $\mathcal{T}_i$ contains only applications of inference rules (no application of RB). The nodes below the root of each 0-depth component $\mathcal{T}_i$ contain only occurrences of A that result from applications of the structural rule RB discharging the virtual assumptions in Δ_i.

[23] By this we mean a notion that can be sensibly associated with the "difficulty" of proving the conclusion from the assumptions both from the computational and the cognitive viewpoint.

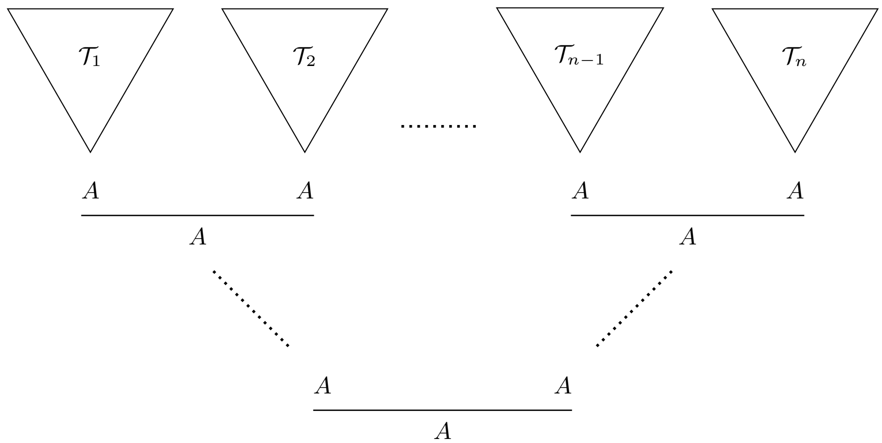

Figure 4: Structure of an RB-canonical C-intelim proof.

Definition 4.4. *Given a C-intelim proof $\mathcal{T}$, we say that an application of RB in $\mathcal{T}$ is* redundant *if one of the following conditions hold:*

1. *at least one of its virtual assumptions is vacuously discharged,*
2. *its conclusion still depends on one of the discharged virtual assumptions.*

Example 4.1. *Consider a C-intelim proof $\mathcal{T}$ containing the following subproof:*

$$\mathcal{T}' = \qquad \dfrac{\dfrac{[A] \quad A \to B}{B} \qquad \dfrac{C \quad C \to B}{B}}{B.}$$

$\mathcal{T}'$ is a proof of B depending on $\Gamma = \{A \to B, C, C \to B\}$ ending with an application of RB in which the virtual assumption $\neg A$ is vacuously discharged in the righthand subproof. Therefore, the last step of this proof is a redundant application of RB. Such redundant application of RB can be eliminated by replacing $\mathcal{T}'$ with:

$$\dfrac{C \quad C \to B}{B.}$$

Example 4.2. *Consider a proof $\mathcal{T}$ containing the following subproof:*

$$\mathcal{T}' = \qquad \dfrac{\dfrac{\dfrac{[A]^1 \quad \neg A}{\curlywedge}}{B} \qquad \dfrac{[\neg A]^2 \quad \neg A \to B}{B}}{B.}\ \mathit{1,2}$$

Here, the application of RB is redundant because its conclusion still depends on the undischarged occurrence of the assumption $\neg A$ in the lefthand subproof. The latter, *however, is discharged as a virtual assumption in the righthand subproof. The redundancy of this application is apparent if we consider that this subproof $\mathcal{T}'$ can be replaced by the following equivalent one (i.e., with the same assumptions and conclusions) that does not contain it:*

$$\frac{\neg A \quad \neg A \rightarrow B}{B}$$

Any C-intelim proof can be turned into one that contains no redundant applications of RB by applying the following transformations[24] that remove the redundant applications of RB:

$$\frac{\begin{array}{c}[A]\\ \mathcal{T}_1\\ B\end{array} \qquad \begin{array}{c}[\neg A]\\ \mathcal{T}_2\\ B\end{array}}{B} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{T}_i\\ B\end{array} \tag{T3}$$

where $i = 1$ if A is vacuously discharged in $\mathcal{T}_1$ or occurs as an undischarged assumption in $\mathcal{T}_2$; $i = 2$ if $\neg A$ is vacuously discharged in $\mathcal{T}_2$ or occurs as an undischarged assumption in $\mathcal{T}_1$.

Note that the result of removing all redundant applications of RB from a C-intelim proof of A depending on Γ will be a C-intelim proof of A depending on $\Delta \subseteq \Gamma$.

Let $d_2(\mathcal{T})$ denote the number of redundant applications of RB in $\mathcal{T}$. Observe that each application of the transformations (T3) yields a tree $\mathcal{T}'$ such that $d_2(\mathcal{T}') < d_2(\mathcal{T})$. Thus:

Lemma 4.2. *Every C-intelim proof of A depending on* Γ *can be turned into a C-intelim proof of A depending on* $\Delta \subseteq \Gamma$ *that contains no redundant applications of RB, by means of any sufficiently long sequence of applications of* (T3) *and with no increase in the size or depth of the proof.*

Remark 4.2. *Observe that every application of* (T3) *that decreases* $d_2(\mathcal{T})$ *does not introduce any new application of RB and so cannot increase* $d_1(\mathcal{T})$*.*

Definition 4.5. *A* detour *in a C-intelim proof* $\mathcal{T}$ *is an occurrence of a formula as conclusion of an introduction and, at the same time, as major premise of an elimination.*

The transformations (T4)–(T16) show how detours can be removed from a C-intelim proof.[25] Note that the final proof will have the same conclusion as the original one and will depend on a subset of the original assumptions. (To save space we use the

[24] See Footnote 21.

[25] The notion of "detour" refers to a redundant use of the intelim rules and the removal of such detours is key in the proof of the normalization theorem. In the context of Gentzen-style natural deduction it corresponds to that of "maximum formula" in [Prawitz, 1965].

variable i ranging over $\{1, 2\}$.)

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_i\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_i\end{matrix}}{A_{(3-i)}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_i\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_i\end{matrix}}{\curlywedge}}{A_{(3-i)}} \tag{T4}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_i\end{matrix}}{A_1 \vee A_2} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_{(3-i)}\end{matrix}}{A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_1\\ A_i\end{matrix} \tag{T5}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_1\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ \neg A_2\end{matrix}}{\neg(A_1 \vee A_2)}}{\neg A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_i\\ \neg A_i\end{matrix} \tag{T6}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix}}{\neg(A_1 \wedge A_2)} \qquad \begin{matrix}\mathcal{T}_2\\ A_i\end{matrix}}{\neg A_{(3-i)}} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ A_i\end{matrix}}{\curlywedge}}{\neg A_{(3-i)}} \tag{T7}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix}}{\neg(A_1 \wedge A_2)} \qquad \begin{matrix}\mathcal{T}_2\\ A_{(3-i)}\end{matrix}}{\neg A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_1\\ \neg A_i\end{matrix} \tag{T8}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A_1\end{matrix} \qquad \begin{matrix}\mathcal{T}_2\\ A_2\end{matrix}}{A_1 \wedge A_2}}{A_i} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_i\\ A_i\end{matrix} \tag{T9}$$

$$\dfrac{\dfrac{\begin{matrix}\mathcal{T}_1\\ A\end{matrix}}{\neg\neg A}}{A} \quad \rightsquigarrow \quad \begin{matrix}\mathcal{T}_1\\ A\end{matrix} \tag{T10}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ A_1\end{array}}{A_2} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ A_1\end{array}}{\curlywedge}}{A_2} \tag{T11}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_2\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ A_1\end{array}}{A_2} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{T}_1\\ A_2\end{array} \tag{T12}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg A_1} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{T}_1\\ \neg A_1\end{array} \tag{T13}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_2\end{array}}{A_1 \to A_2} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg A_1} \quad \rightsquigarrow \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_2\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\curlywedge}}{\neg A_1} \tag{T14}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_1\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg(A_1 \to A_2)}}{A_1} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{T}_1\\ A_1\end{array} \tag{T15}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ A_1\end{array} \qquad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array}}{\neg(A_1 \to A_2)}}{\neg A_2} \quad \rightsquigarrow \quad \begin{array}{c}\mathcal{T}_2\\ \neg A_2\end{array} \tag{T16}$$

Note that (i) the transformations (T4)–(T16) never increase the size of the proof, nor do they increase its depth; (ii) in some cases, their application may introduce new detours; for example in (T4) if A_i or $\neg A_i$ are conclusions of introductions in $\mathcal{T}_1$ or $\mathcal{T}_2$, they are not detours in the original subproof, but become such in the transformed one. However these new detours are always of lower complexity than the one that is removed by the transformation.

So, let $d_3(\mathcal{T})$ be the sum of the complexities (i.e., number of the logical operators) of all detours occurring in $\mathcal{T}$ and equal to 0 when there are no detours. Then, each application of the transformations (T4)–(T16) decreases the value of $d_3(T)$ until it drops to 0, so yielding a proof that is detour-free.

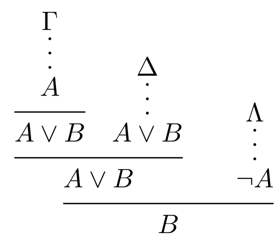

Figure 5: Higher level detours

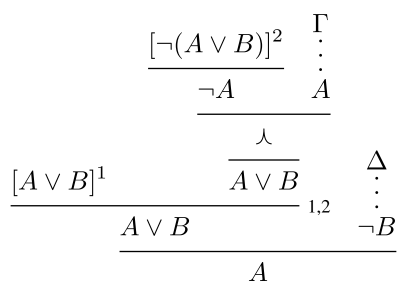

Figure 6: Indirect detours

Lemma 4.3. *Any C-intelim proof of A depending on Γ can be transformed into a C-intelim proof of A depending on $\Delta \subseteq \Gamma$ that contains no detours, by any sufficiently long sequence of applications of* (T4)–(T16) *and with no increase in the size or depth of the proof.*

Remark 4.3. *Note that the transformations* (T4)–(T16) *do not introduce any new application of RB, and therefore cannot increase either $d_1(\mathcal{T})$ or $d_2(\mathcal{T})$.*

Remark 4.4. *We could generalize the notion of "detour" by considering any sequence $A_1, \ldots, A_n$ of occurrences of the* same formula *such that: (i) A_1 is the conclusion of an introduction, (ii) A_n is the major premise of an elimination, (iii) for all i such that $1 < i \leq n$, A_i is an immediate successor of A_{i-1} resulting from an application of RB. Such as sequence is a* detour of level n*. An example of detour of level 2 is given in Figure 5. However, observe that such higher level detours* cannot occur in RB-canonical proofs*, for A_n would be at the same time the conclusion of an application of RB and a premise of an inference rule. Hence, elimination of higher level detours is a side-effect of transforming proofs into RB-canonical ones.*

Definition 4.6. *Given a C-intelim proof $\mathcal{T}$, an application of RNC is* canonical *in $\mathcal{T}$ if it is not the case that its premises are both conclusions of introductions. A C-intelim proof is* RNC-canonical *if it contains no non-canonical applications of RNC.*

Non-canonical applications of RNC can be removed by means of the the transformations T17–T20 (for $i = 1, 2$).

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ A_i\end{array}}{A_1 \vee A_2}\qquad\dfrac{\begin{array}{cc}\mathcal{T}_1&\mathcal{T}_2\\ \neg A_1&\neg A_2\end{array}}{\neg(A_1 \vee A_2)}}{\curlywedge}\quad\rightsquigarrow\quad\frac{\begin{array}{cc}\mathcal{T}_0&\mathcal{T}_i\\ A_i&\neg A_i\end{array}}{\curlywedge}\tag{T17}$$

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ \neg A_i\end{array}}{\neg(A_1 \wedge A_2)}\qquad\dfrac{\begin{array}{cc}\mathcal{T}_1&\mathcal{T}_2\\ A_1&A_2\end{array}}{A_1 \wedge A_2}}{\curlywedge}\quad\rightsquigarrow\quad\frac{\begin{array}{cc}\mathcal{T}_i&\mathcal{T}_0\\ A_i&\neg A_i\end{array}}{\curlywedge}\tag{T18}$$

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ \neg A_1\end{array}}{A_1 \to A_2}\qquad\dfrac{\begin{array}{cc}\mathcal{T}_1&\mathcal{T}_2\\ A_1&\neg A_2\end{array}}{\neg(A_1 \to A_2)}}{\curlywedge}\quad\rightsquigarrow\quad\frac{\begin{array}{cc}\mathcal{T}_1&\mathcal{T}_0\\ A_1&\neg A_1\end{array}}{\curlywedge}\tag{T19}$$

$$\frac{\dfrac{\begin{array}{c}\mathcal{T}_0\\ A_2\end{array}}{A_1 \to A_2}\qquad\dfrac{\begin{array}{cc}\mathcal{T}_1&\mathcal{T}_2\\ A_1&\neg A_2\end{array}}{\neg(A_1 \to A_2)}}{\curlywedge}\quad\rightsquigarrow\quad\frac{\begin{array}{cc}\mathcal{T}_0&\mathcal{T}_2\\ A_2&\neg A_2\end{array}}{\curlywedge}\tag{T20}$$

By the *complexity of an application of RNC* with premises A and $\neg A$, we mean the logical complexity of A. Again, the removal of a non-canonical application of RB may introduce a new non-canonical application. but the complexity of the latter is always lower. So, let $d_4(\mathcal{T})$ be sum of the complexities of the non-canonical applications of RNC in $\mathcal{T}$ and equal to 0 when all the applications of RNC are canonical. Each of the transformations (T17)–(T20) decreases $d_4(\mathcal{T})$ until its value drops to 0.

Lemma 4.4. *Any C-intelim proof of A depending on Γ can be transformed into an RNC-canonical C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by means of any sufficiently long sequence of applications of* (T17)–(T20) *and with no increase in the size or depth of the proof.*

Remark 4.5. *Observe that applications of* (T17)–(T20) *do not introduce new applications of RB, nor do they introduce new detours*[26]*, and so cannot increase any of the parameters $d_1(\mathcal{T})$–$d_3(\mathcal{T})$.*

Definition 4.7. *Given a C-intelim proof $\mathcal{T}$, an application of XFQ is* canonical *in $\mathcal{T}$ if (i) its conclusion is not $\curlywedge$, and (ii) its conclusion is not the premise of an application of an inference rule. A C-intelim proof is* XFQ-canonical *if it contains no non-canonical applications of XFQ.*

[26]Recall that in C-intelim RNC is a structural rule and not an elimination rule; the rules for unsigned formulae are intended as practical proxies for their signed versions. See the discussion in Section 3.

Remark 4.6. *Observe that if a proof of A depending on Γ, is XFQ-canonical, it may contain only applications of XFQ with A itself as conclusion. If it is also RB-canonical, applications of XFQ may occur only as the last step in one of its 0-depth components.*

The notion of XFQ-canonical proof makes it apparent that the only use of XFQ consists, in fact, in showing that the assumptions are inconsistent, given that this rule can be applied only as the last step of a 0-depth component that immediately follows an application of RNC.

Any C-intelim proof can be turned into one that is XFQ-canonical by repeatedly applying the following transformations:

$$\dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{\curlywedge} \quad\rightsquigarrow\quad \begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array} \tag{T21}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{C}}{D} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{D} \tag{T22}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{C} \quad \begin{array}{c}\mathcal{T}_2\\ D\end{array}}{E} \quad\rightsquigarrow\quad \dfrac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{E} \tag{T23}$$

As before, the transformations (T22) and (T23) can be applied, respectively, for any instance C/D of a one-premise inference rule, any instance $C, D/E$ of a two-premise inference rule. They eventually yield a proof in which the conclusion of XFQ is never used as premise of an intelim rule or of RNC. This implies that all applications of XFQ are followed only by applications of RB. Note that the final proof will have the same conclusion as the original one and depend on a subset of the original assumptions.

Given a subproof $\mathcal{T}'$ of $\mathcal{T}$ ending with an application of XFQ, let $\mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$ be the number of occurrences of formulae below the root of $\mathcal{T}'$ that result from applications of inference rules. Then, we define $d_5(\mathcal{T})$ as follows:

$$d_5(\mathcal{T}) \sum_{T' \in S} \mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}') \tag{7}$$

where S is the set of subproofs of $\mathcal{T}$ ending with an application of XFQ. Observe that each application of the transformations (T22)–(T23) yields a tree $\mathcal{T}'$ such that $d_5(\mathcal{T}') < d_5(\mathcal{T})$. Thus:

Lemma 4.5. *Any C-intelim proof of A depending on Γ can be transformed into an XFQ-canonical C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by any sufficiently long sequence of applications of* (T22)–(T23) *and with no increase in the size or depth of the proof.*

DEFINITIONS OF d_1–d_5
$d_1(\mathcal{T}) = \sum_{T' \in S} \mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}')$ where S is the set of *distinct* subproofs of $\mathcal{T}$ ending with an application of RB and $\mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}')$ is the number of occurrences of formulae below $\mathcal{T}'$'s root that result from applications of inference (intelim or falsum) rules
$d_2(\mathcal{T})$ = number of redundant applications of RB in $\mathcal{T}$
$d_3(\mathcal{T})$ = 0 if there are no detours and the sum of the complexities (i.e., number of the logical operators) of all detours occurring in $\mathcal{T}$ otherwise
$d_4(\mathcal{T})$ = 0 if all the applications of RNC in $\mathcal{T}$ are canonical and the sum of the complexities of the non-canonical applications of RNC in $\mathcal{T}$ otherwise
$d_5(\mathcal{T}) \sum_{T' \in S} \mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$ where S is the set of subproofs of $\mathcal{T}$ ending with an application of XFQ and $\mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$ is the number of occurrences of formulae below $\mathcal{T}'$'s root that result from applications of inference rules.

Table 4: Transformation parameters.

Remark 4.7. *Observe that any application of* (T22)–(T23) *does not introduce new applications of RB; nor does it introduce new detours or new non-canonical applications of RNC, and so cannot increase any of the parameters* $d_1(\mathcal{T})$*–*$d_4(\mathcal{T})$*.*

Definition 4.8. *We say that a C-intelim proof* $\mathcal{T}$ *is* quasi-normal *if all of the following conditions are satisfied:*

- $\mathcal{T}$ *is RB-canonical, RNC-canonical and XFQ-canonical,*
- $\mathcal{T}$ *contains no redundant applications of RB,*
- $\mathcal{T}$ *contains no detours.*

The definitions of $d_1(\mathcal{T})$–$d_5(\mathcal{T})$ given above are summarized in Table 4. Now, let

$$d(T) = \langle d_1(\mathcal{T}), d_2(T), d_3(\mathcal{T}), d_4(\mathcal{T}), d_5(\mathcal{T}) \rangle$$

and consider the usual lexicographic order on $d(\mathcal{T})$ for all C-intelim proofs $\mathcal{T}$. The reader can verify that a transformation that decreases $d_i(\mathcal{T})$ for some $i < 5$, may increase $d_j(\mathcal{T})$ for some $j > i$. However, as observed in Remarks 4.2–4.7, transformations that decrease $d_i(\mathcal{T})$ for $i > 1$, never increase d_j for any $j < i$. So, each of the transformations (T1)–(T23) decreases $d(\mathcal{T})$. Hence the repeated application of these transformations, independently of their order, eventually yields a proof $\mathcal{T}'$ such that $d(\mathcal{T}') = \langle 0, 0, 0, 0, 0 \rangle$, which is therefore quasi-normal.

Theorem 4.1. *Any C-intelim proof of A depending on Γ can be turned into a quasi-normal C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by means of any sufficiently long sequence of applications of the transformations* (T1)–(T23).

Moreover, observe that the transformations d_2–d_5 never increase the size of the proof and do not introduce any new application of RB. Hence,

Theorem 4.2. *Any RB-canonical C-intelim proof of A depending on Γ can be turned into a quasi-normal C-intelim proof of A depending on $\Delta \subseteq \Gamma$, by means of any sufficiently long sequence of applications of the transformations* (T3)–(T20) *and with no increase in the size or depth of the proof.*

Remark 4.8. *If $\mathcal{T}$ is a quasi-normal C-intelim proof,*

- *every subproof of $\mathcal{T}$ is also quasi-normal;*
- *every 0-depth component of $\mathcal{T}$ contains at most one application of RNC and at most one application of XFQ (as the last step).*

5 Intermediate conclusions

In this first part we have introduced the C-intelim system and argued that it is better suited than Gentzen-style natural deduction to represent naturally the inferences of classical logic. The main technical contribution of this part is a set of transformations that can turn, independently of the order in which they are applied, any C-intelim proof into a *quasi-normal* one. What have we achieved through the notion of quasi-normal proof? Not only is this notion a convenient step towards the normalization result that will be presented in Part II, but it is also interesting in its own right. Quasi-normal proofs avoid trivially redundant applications of RB and trivially redundant applications of inference rules (detours). Moreover, the applications of RB are pushed down at the end of the proof-tree, so that their conclusion is always the conclusion of the whole proof and their role consists in gradually discharging the virtual assumptions made in the properly inferential components that we called "0-depth components" (see Figure 4). In this way we make a clear separation between the applications of inference rules and the application of RB that allows us to define the depth of an argument in a straightforward way, although this notion will show its full significance only in Part II, when the notion of normal proof will be introduced. Furthermore, in each 0-depth component, RNC is applied at most once, and also XFQ is applied at most once as the last step, allowing us to "infer", if only in a pickwickian sense, the conclusion of the whole proof as a result of the inconsistency of the assumptions.

In the Gentzen tradition, it is part of the logical folklore to identify *analytic* proofs — i.e., proofs that enjoy the (weak) subformula property — with normal proofs and the latter with proofs that contain no "detours" – i.e., no obviously redundant sequences of applications of inference rules. However, it is well-known that, for some infinite sets of classically valid inferences, analytic proofs can be exponentially longer than non-analytic ones, e.g., proofs in Frege systems or in the sequent calculus with cut. Here we clearly distinguish between analyticity on the one hand and absence of detours on the other. Quasi-normal proofs are proofs that, albeit being *non-analytic*, can be legitimately regarded as containing no trivial "detour". It is easy to see that such proofs can polynomially simulate Frege system (see Section 5 of Part II), for which the existence of a polynomial upper bound on proof length has not been disproved yet. Hence, as far as the length of proofs is concerned, the restriction of C-intelim to quasi-normal proofs is among the most powerful proof systems for classical propositional logic. As we shall see in Part II, the set of formulae that can be used as RB-formulae, that we have called the *virtual space* in the comment to Definition 4.2, may be bounded in a variety of ways without loss of completeness. The strictest way of bounding it when generating a proof of A from Γ consists in allowing as RB-formulae only atomic formulae that occur in $\Gamma \cup \{A\}$. A more liberal restriction consists in allowing only subformulae of the formulae in $\Gamma \cup \{A\}$. Shorter proofs can be obtained by further liberalizing the composition of the virtual space allowing for proofs that do not enjoy the subformula property, but in which the virtual space is still bounded. In quasi-normal proofs RB is, at the same time, the only rule that may bring about violations of the subformula property and the only rule that increases the depth of the reasoning process. In C-intelim the transition from analytic proofs to non-analytic (possibly shorter) ones depends only on the way in which we bound the virtual space, i.e. on the set of formulae that are allowed as RB-formulae. Once the virtual space is suitably bounded, the transition from easier proofs to more difficult ones depends only on the depth at which applications of RB are required.

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