Normality, Non-Contamination and Logical Depth in Classical Natural Deduction
Technical Report, Part II

Marcello D'Agostino[1], Dov Gabbay[2], and Sanjay Modgil[3]

[1]Università degli Studi di Milano
[2]Bar Ilan University, King's College London and University of Luxembourg
[3]King's College London

27 March 2018

1 Normal proofs

As mentioned in the general introductory section of Part I, for Gentzen the importance of the normalization theorem consisted mainly in the fact that normal proofs enjoy the *subformula property* to the effect that in the search for proofs we can restrict our attention to inference steps whose conclusion is a "component" either of the assumptions or of the conclusion. This involves a drastic reduction of the search space that is crucial for the purpose of automated deduction. In the case of propositional logic, this search space is finite for each putative inference, paving the way for decision procedures. We know now that "analytic proofs", i.e., those that enjoy the subformula property are not always shorter than non-analytic ones, and may be exponentially longer for certain infinite classes of inferences. For this reason we attached a special importance, in Part I, to quasi-normal proofs that do not enforce the subformula property while still avoiding trivial detours. We now turn to the notion of *normal* proof, which is simply a quasi-normal proof in which RB is applied only to (weak) subformulae either of the premises or of the conclusion. Given this restriction, the resulting proof will be shown to enjoy the subformula property.

For every formula A, a *subformula* of A is defined inductively as follows: (i) A is a subformula of A, (ii) for every binary operator $\circ$, if $B \circ C$ is a subformula of A, then so are B and C, (iii) if $\neg B$ is a subformula of A, then so is B; (iv) nothing else is a subformula of A. A is a *proper subformula* of B if A is a subformula of B, but B is not a subformula of A. A is an *immediate proper subformula* of B, if A is a proper subformula of B and it is not a proper subformula of any proper subformula of B. We shall often say just *immediate subformula* as an abbreviation for "immediate proper subformula".

We say that B is a *weak subformula* of A if either B is a subformula of A or $B = \neg C$ for some subformula C of A.

Remark 1.1. *Notice that the relation "A is a weak subformula of B" is not, in general, transitive. A simple counterexample is given by the triple $\neg\neg A, \neg A, A$.*

Finally, we say that A is a *proper weak subformula* of B if A is a weak subformula of B, but B is not a weak subformula of A. For example, A is a weak subformula of $\neg A$, but not a proper weak subformula; $\neg A$ is a weak subformula of A and of $\neg\neg A$, but not a proper weak subformula of either. On the other hand, both A and $\neg A$ are proper weak subformulae of $A \wedge B$, of $\neg A \wedge B$, of $\neg(A \wedge B)$, etc.

Remarks 1.2. *Observe that:*

1. *The relation "A is a proper weak subformula of B" is transitive: if A is a proper weak subformula of B and B is a proper weak subformula of C, then A is a proper weak subformula of C;*
2. *the minor premise of an elimination is always a proper weak subformula of its major premise;*
3. *the conclusion of an elimination is always a proper weak subformula of its major premise;*
4. *a premise of an introduction is always a proper weak subformula of its conclusion;*

Recall that C is an *RB-formula of* a C-intelim tree $\mathcal{T}$ if C and $\neg C$ are the virtual assumptions discharged by some application of RB in $\mathcal{T}$.

Definition 1.1. *Given a C-intelim proof $\mathcal{T}$ of A depending on Γ, we say that an application of RB in $\mathcal{T}$ is* analytic *if its RB-formula is a subformula of some formula in $\Gamma \cup \{A\}$ (or, equivalently, both the virtual assumptions discharged by this application are weak subformulae of some formula in $\Gamma \cup \{A\}$).*

Definition 1.2. *We say that an application of RB is* atomic *if its RB-formula is atomic, i.e., the virtual assumptions discharged by it have the form p and $\neg p$ for some atomic p.*

Definition 1.3. *A C-intelim proof $\mathcal{T}$ is* normal *if it is quasi-normal and every application of RB in $\mathcal{T}$ is analytic. It is* atomically normal *if it is normal and every application of RB in $\mathcal{T}$ is atomic.*

Note that a quasi-normal 0-depth C-intelim proof, i.e., one that contains no applications of RB, is by definition (atomically) normal.

Remark 1.3. *If $\mathcal{T}$ is a(n atomically) normal C-intelim proof, every subproof of $\mathcal{T}$ is also (atomically) normal.*

Definition 1.4. *We say that a C-intelim proof $\mathcal{T}$ of A depending on Γ has the* weak subformula property *(WSFP) if every formula occurring in $\mathcal{T}$ is either an occurrence of $\curlywedge$ or a weak subformula of some formula in $\Gamma \cup \{A\}$.*

Theorem 1.1. *If $\mathcal{T}$ is a normal 0-depth proof of A depending on Γ, and B is a formula occurring in $\mathcal{T}$, then either $B \in \Gamma \cup \{A\}$, or $B = \curlywedge$, or B is a proper weak subformula of some formula in $\Gamma \cup \{A\}$.*

Proof. Suppose $\mathcal{T}$ is a normal 0-depth proof of A depending on Γ. Let Δ be the set of all formulae C occurring in $\mathcal{T}$ such that (i) $C \notin \Gamma \cup \{A\}$, (ii) $C \neq \curlywedge$, and (iii) C is not a proper weak subformula of any formula in $\Gamma \cup \{A\}$. Let us assume that $\Delta \neq \emptyset$ and take a formula D in Δ of maximal complexity. Since $D \notin \Gamma$, D occurs in $\mathcal{T}$ as conclusion of an application of an intelim rule or of a falsum rule. It cannot be the conclusion of an application of RNC, otherwise $D = \curlywedge$ and $\curlywedge \notin \Delta$ by definition of Δ. Moreover, since $\mathcal{T}$ is normal, it is XFQ-canonical. Thus, D cannot be the conclusion of an application of XFQ, otherwise it should be equal to the conclusion A of $\mathcal{T}$, which does not belong to Δ by definition of Δ.

Furthermore, D cannot be the conclusion of an elimination. To see this, observe that the major premise of this elimination cannot be in $\Gamma \cup \{A\}$, otherwise, by Remark 1.2.3, D would be a proper weak subformula of a formula in $\Gamma \cup \{A\}$ and therefore would not belong to Δ. Moreover, this major premise cannot be a proper weak subformula of a formula in $\Gamma \cup \{A\}$, because in this case, by Remarks 1.2.1 and 1.2.3, D would also be a proper weak subformula of some formula in $\Gamma \cup \{A\}$ and therefore would not belong to Δ. Hence the major premise of the elimination should be a formula in Δ of greater complexity than D, against the assumption that D is a formula of maximal complexity in Δ.

Hence, D can only be the conclusion of an introduction. Since $D \neq A$, D must be used in $\mathcal{T}$ as premise of some intelim rule or of some of the falsum rules. Since $D \neq \curlywedge$, it cannot be used as premise of XFQ. Moreover, it cannot be used as major premise of an elimination rule, otherwise D would be a detour, against the assumption that $\mathcal{T}$ is normal (and so contains no detours). Furthermore, it cannot be used as premise of RNC, because in this case the complementary premise, call it D', could also be only the conclusion of an introduction, for the same reasons as D; but this is impossible because $\mathcal{T}$ is normal and therefore RNC-canonical (it is not the case that both the premises of an application of RNC are both conclusions of an introduction). Finally, it cannot be used as minor premise of an elimination, otherwise, by Remark 1.2.2, D would be a proper weak subformula of the major premise. So, either this major premise belongs to $\Gamma \cup \{A\}$ and then, by Remark 1.2.1, D would be a proper weak subformula of some formula in $\Gamma \cup \{A\}$, in which case D would not belong to Δ; or the major premise of this elimination would be a formula in Δ of greater complexity than D, against the assumption that D is a formula in Δ of maximal complexity .

Thus, D, must be used as premise of an introduction. But this is impossible, because, by Remark 1.2.4, D would be a proper weak subformula of the conclusion of this introduction. So, either this conclusion belongs to $\Gamma \cup \{A\}$ and, by Remark 1.2.1, D would be a proper weak subformula of some formula in $\Gamma \cup \{A\}$, in which case D would not belong to Δ, or the conclusion of this introduction would be a formula in Δ of greater complexity than D, against the assumption that D is a formula in Δ of maximal complexity . Hence, Δ must be empty. □

The above lemma immediately implies the following:

Corollary 1.1 (WSFP of 0-depth deductions)**.** *Every normal 0-depth C-intelim proof has the WSFP.*

Remark 1.4. *Note that if $\mathcal{T}$ is a quasi-normal proof of A depending on Γ, whose 0-depth components are $\mathcal{T}_1, \ldots, \mathcal{T}_n$, every 0-depth component $\mathcal{T}_i$ is a normal proof of A depending on $\Gamma_i \cup \Delta_i$, where $\Gamma_i \subseteq \Gamma$ and Δ_i are virtual assumptions that are subsequently discharged in $\mathcal{T}$ by applications of RB.*

By virtue of the exclusion of detours, the structure of each 0-depth component of a normal C-intelim proof is determined quite sharply.

Definition 1.5. *Given a 0-depth proof $\mathcal{T}$ of A depending on Γ, an* intelim walk *of $\mathcal{T}$ is a sequence $A_1, \ldots, A_n$ of occurrences of formulae such that (i) $A_1 \in \Gamma$, (ii) for $1 \leq i < n$, A_i is a premise of an application of an intelim rule with A_{i+1} as conclusion; (iii) A_n is either the conclusion A of $\mathcal{T}$, or the minor premise of an elimination or a premise of an application of RNC.*

Definition 1.6. *If $\mathcal{T}$ is a 0-depth proof of A depending on Γ, a* normal intelim walk *of $\mathcal{T}$ is an intelim walk of the form*

$$A_1, \ldots, A_m, \ldots, A_{m+n},$$

with $m \geq 1$ and $n \geq 0$, where

1. *if $m > 1$, for $i < m$, A_i is the major premise of an elimination and the subsequence $A_1, \ldots, A_m$ is called the* E-part *of the intelim walk; if $m = 1$, we say that the E-part is empty;*
2. *if $n > 0$, for $m < i \leq m + n$, A_i is the conclusion of an introduction and the subsequence $A_m, \ldots A_{m+n}$ is called the* I-part *of the intelim walk; if $n = 0$, we say that the I-part is empty;*
3. *A_m is called the* minimum formula *of the normal intelim walk.*

A normal intelim walk is complete *if its last formula is either a premise of RNC or the conclusion A of the proof.*

Example 1.1. *Both the immediate subproofs of the C-intelim proof In Figure 1 of Part I are 0-depth proofs. The normal intelim walks of the lefthand subproof are the following sequences: A (two copies, both the E-part and the I-part are empty); $A \to \neg B, \neg B$ (I-part empty); $B \vee C, C$ (I-part empty); $\neg(C \wedge \neg B), \neg\neg B, B$ (I-part empty, complete); $A \to \neg B, \neg B$ (I-part empty, complete). In the right-hand subproof the normal intelim walks are: $\neg G \to D, \neg\neg G, G$ (I-part empty, complete); $E \vee F \to \neg D, \neg D$ (I-part empty); $\neg A$ (both E-part and I-part empty); $A \vee E, E, E \vee F$. In each of the C-intelim proofs of Figure 2 of part I, both immediate subproofs are 0-depth; the left branch of the first subproof is a complete normal intelim walk and its minimum formula is $\neg B$; the only branch of the second subproof is a normal intelim walk whose E-part is empty and its minimum formula is $\neg A$.*

The proof of the following lemma is left to the reader.[1]

[1] Note that what we call "branch" in this paper is called "thread" in [Prawitz, 1965].

Lemma 1.1. *If $\mathcal{T}$ is a 0-depth normal proof, then*

- *every branch of $\mathcal{T}$ contains a normal intelim walk;*
- *at least one branch of $\mathcal{T}$ contains a complete normal intelim walk;*
- *the minimum formula of a normal intelim walk is a weak subformula of all the formulae in it; if the E-part is non-empty, the minimum formula is a proper weak subformula of all the formulae preceding it in the walk; if the I-part is non-empty the minimum formula is a proper weak subformula of all the formulae following it in the walk.*

Theorem 1.2 (WSFP of normal deductions)**.** *If $\mathcal{T}$ is a normal C-intelim proof, then $\mathcal{T}$ has the WSFP.*

Proof. Let $\mathcal{T}_1 \ldots, \mathcal{T}_n$ be the 0-depth components of $\mathcal{T}$. By Remark 1.4, every 0-depth component of $\mathcal{T}$ is normal. Recall that in a normal proof, every formula occurring in $\mathcal{T}$ occurs also in some of its 0-depth components, since all the conclusions of applications of RB are equal to the conclusion of all 0-depth components. Thus, for every formula B occurring in $\mathcal{T}$, there is a 0-depth component $\mathcal{T}_i$ of $\mathcal{T}$ such that, by Theorem 1.1, either B is in $\Gamma_i \cup \Delta_i \cup \{A\}$ or $B = \curlywedge$, or B is a proper weak subformula of some formula in $\Gamma_i \cup \Delta_i \cup \{A\}$, where Γ_i are the assumptions of $\mathcal{T}_i$ that are left undischarged in $\mathcal{T}$ and Δ_i are the virtual assumptions subsequently discharged in $\mathcal{T}$. If $\mathcal{T}$ is normal, every formula in Δ_i is a weak subformula of a formula in $\Gamma_i \cup \{A\}$. So, either (i) $B = \curlywedge$, or (ii) $B \in \Gamma_i \cup \Delta_i \cup \{A\}$ and so B is a weak subformula of a formula in $\Gamma_i \cup \{A\}$, or (iii) B is a *proper* weak subformula of some formula in $\Gamma_i \cup \Delta_i \cup \{A\}$ and, since all the formulae in Δ_i are weak subformulae of some formula in $\Gamma_i \cup \{A\}$. It is not difficult to verify, that if B is a proper weak subformula of C and C is a weak subformula of D, then B is a weak subformula of D. Hence, B must be a weak subformula of some formula in $\Gamma_i \cup \{A\}$. $\square$

We shall now show that any C-intelim proof can be turned into an (atomically) normal one. First, observe that any C-intelim proof can be turned into one in which all applications of RB are *either* analytic or *atomic* (inclusive or) by repeatedly applying the following transformations:[2]

$$\dfrac{\begin{matrix}[A\vee B]^1 & [\neg(A\vee B)]^2\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}\,{}^{1,2} \quad\leadsto\quad \dfrac{\begin{matrix}\dfrac{[A]^1}{A\vee B}\\ \mathcal{T}_1\\ C\end{matrix} \qquad \dfrac{\begin{matrix}\dfrac{[B]^3}{A\vee B} & \dfrac{[\neg A]^2 \quad [\neg B]^4}{\neg(A\vee B)}\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}\,{}^{3,4}}{C}\,{}^{1,2} \qquad \text{(T28)}$$

[2] The reader can compare them to the ones used in [Tennant, 1990], pp. 95–96.

$$
\dfrac{\begin{matrix}[A\wedge B]^1 & [\neg(A\wedge B)]^2\\ \mathcal{T}_1 & \mathcal{T}_2 \\ C & C\end{matrix}}{C}{\scriptstyle 1,2}
\quad\rightsquigarrow\quad
\dfrac{\dfrac{\begin{matrix}\dfrac{[A]^1\quad [B]^3}{A\wedge B} & \dfrac{[\neg B]^4}{\neg(A\wedge B)}\\ \mathcal{T}_1 & \mathcal{T}_2\\ C & C\end{matrix}}{C}{\scriptstyle 3,4}\quad \begin{matrix}\dfrac{[\neg A]^2}{\neg(A\wedge B)}\\ \mathcal{T}_2\\ C\end{matrix}}{C}{\scriptstyle 1,2}
\tag{T29}
$$

$$
\dfrac{\begin{matrix}[A\to B]^1 & [\neg(A\to B)]^2\\ \mathcal{T}_1 & \mathcal{T}_2 \\ C & C\end{matrix}}{C}{\scriptstyle 1,2}
\quad\rightsquigarrow\quad
\dfrac{\begin{matrix}\dfrac{[B]^1}{A\to B}\\ \mathcal{T}_1\\ C\end{matrix}\quad \dfrac{\begin{matrix}\dfrac{[A]^3\quad [\neg B]^2}{\neg(A\to B)} & \dfrac{[\neg A]^4}{A\to B}\\ \mathcal{T}_2 & \mathcal{T}_1\\ C & C\end{matrix}}{C}{\scriptstyle 3,4}}{C}{\scriptstyle 1,2}
\tag{T30}
$$

$$
\dfrac{\begin{matrix}[\neg A]^1 & [\neg\neg A]^2\\ \mathcal{T}_1 & \mathcal{T}_2 \\ B & B\end{matrix}}{B}{\scriptstyle 1,2}
\quad\rightsquigarrow\quad
\dfrac{\begin{matrix}\dfrac{[A]^1}{\neg\neg A} & [\neg A]^2\\ \mathcal{T}_2 & \mathcal{T}_1\\ B & B\end{matrix}}{B}{\scriptstyle 1,2}
\tag{T31}
$$

Note that, in general, these transformations increase the size of the proof. Moreover, they may introduce new detours; for example in (T28) it may be the case that $A \vee B$ or $\neg(A \vee B)$ or both are used in $\mathcal{T}_1$ or $\mathcal{T}_2$ as major premises of eliminations. They may also introduce new non-canonical applications of RNC.

Now, for every C-intelim tree $\mathcal{T}$ let $g(\mathcal{T})$ be defined as follows:

$$
g(\mathcal{T}) = \begin{cases} \sharp(A) & \text{if } \mathcal{T} \text{ ends with a non-analytic application of RB and } A \text{ is the} \\ & \text{RB-formula of this application} \\ 0 & \text{otherwise,} \end{cases} \tag{1}
$$

where $\sharp(A)$ denotes the logical complexity of (the total number of occurrences of logical operators in) A.

Let

$$
d_0(\mathcal{T}) = \sum_{\mathcal{T} \in \mathrm{SUB}(\mathcal{T})} g(\mathcal{T}). \tag{2}
$$

where $\mathrm{SUB}(\mathcal{T})$ is the set of all subtrees of $\mathcal{T}$. Each application of the transformations (T28)-(T31) decreases $d_0(\mathcal{T})$ until its value drops to 0, which means that all the applications or RB are either analytic or atomic.

Lemma 1.2. *Any C-intelim proof of A depending on Γ can be transformed into a C-intelim proof of A depending on $\Delta \subseteq \Gamma$ where all the applications of RB are either analytic or atomic.*

Then we show the following:

Lemma 1.3. *If $\mathcal{T}$ is a quasi-normal C-intelim proof of A and all the non-atomic applications of RB in $\mathcal{T}$ are analytic, then all the atomic applications of RB in $\mathcal{T}$ are also analytic, i.e. $\mathcal{T}$ is normal.*

Proof. Let $\mathcal{T}$ be a quasi-normal C-intelim proof of A depending on Γ such that all the non-atomic applications of RB in $\mathcal{T}$ are analytic. We now show that $\mathcal{T}$ cannot contain any non-analytic atomic applications of RB and, therefore, $\mathcal{T}$ is normal. For this purpose we prove, by induction on k, that every k-depth subproof of $\mathcal{T}$ is normal.

By Remark 1.4, every 0-depth subproof of $\mathcal{T}$ is normal. For $k > 0$, assume that every subproof of $\mathcal{T}$ of depth k–1 is normal. We show that under this assumption every k-depth subproof is also normal. Since $k > 0$, any k-depth subproof either ends with a non-atomic application of RB (which is by hypothesis analytic, so that the proof is normal) or has the following form, for some atomic p:

$$\dfrac{\begin{matrix}[p]^1 \\ \mathcal{T}_1 \\ A\end{matrix} \qquad \begin{matrix}[\neg p]^2 \\ \mathcal{T}_2 \\ A\end{matrix}}{A.}\,1,2 \tag{3}$$

Suppose, ex absurdo, that this atomic application of RB is non-analytic, i.e. p that does not occur in $\Gamma \cup \{A\}$. By inductive hypothesis, we know that for some $\Gamma_1, \Gamma_2 \subseteq \Gamma$:

- $\mathcal{T}_1$ is a normal proof of A from $\Gamma_1 \cup \Lambda_1 \cup \{p\}$,
- $\mathcal{T}_2$ is a normal proof of A from $\Gamma_2 \cup \Lambda_2 \cup \{\neg p\}$,

where Λ_1 and Λ_2 are the sets of virtual assumptions that are still undischarged in $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively.

Since $\mathcal{T}$ is quasi normal, it contains no redundant applications of RB, and so neither p nor $\neg p$ are vacuously discharged in (3). Thus, p must be used as premise of some application of an inference rule in $\mathcal{T}_1$ and $\neg p$ as premise of some application of an inference rule in $\mathcal{T}_2$. By its logical form, p cannot be used in $\mathcal{T}_1$ as major premise of an elimination. If it is used as minor premise of an elimination, p would occur in the major premise and the latter would not be a weak subformula of some formula in $\Gamma_1 \cup \Lambda_1 \cup \{p\} \cup \{A\}$, for every formula in Λ_1 is either an atomic formula, or the negation of an atomic formula, or is a weak subformula of some formula in $\Gamma_1 \cup \{A_1\}$. But this is impossible, since, by inductive hypothesis, $\mathcal{T}_1$ is normal and, by Theorem 1.2, it has the WSFP.

Moreover, p cannot be used as premise of an introduction, for the conclusion of this introduction would not be a weak subformula of any of the formulae in $\Gamma_1 \cup \Lambda_1 \cup \{p\} \cup \{A\}$, which would again contradict the hypothesis that $\mathcal{T}_1$ is normal and has the WSFP. So, p must be used in $\mathcal{T}_2$ as premise of some application of the falsum rules. It is impossible that p is used as premise of XFQ, for we have stipulated that $\curlywedge$ cannot occur in the assumptions. Hence, p can be used only as premise of an application of RNC in $\mathcal{T}_1$. But, in this case, the other premise $\neg p$ cannot result from the application of an elimination, otherwise $\mathcal{T}_1$ would not have the WSFP, against the hypothesis that

DEFINITIONS OF d_1–d_5

$d_1(\mathcal{T}) = \sum_{T' \in S} \mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}')$

where S is the set of *distinct* subproofs of $\mathcal{T}$ ending with an application of RB and $\mathrm{dnc}_{\mathrm{RB}}(\mathcal{T}')$ is the number of occurrences of formulae below $\mathcal{T}'$'s root that result from applications of inference (intelim or falsum) rules

$d_2(\mathcal{T})$ = number of redundant applications of RB in $\mathcal{T}$

$d_3(\mathcal{T})$ = 0 if there are no detours and the sum of the complexities (i.e., number of the logical operators) of all detours occurring in $\mathcal{T}$ otherwise

$d_4(\mathcal{T})$ = 0 if all the applications of RNC in $\mathcal{T}$ are canonical and the sum of the complexities of the non-canonical applications of RNC in $\mathcal{T}$ otherwise

$d_5(\mathcal{T}) \sum_{T' \in S} \mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$

where S is the set of subproofs of $\mathcal{T}$ ending with an application of XFQ and $\mathrm{dnc}_{\mathrm{XFQ}}(\mathcal{T}')$ is the number of occurrences of formulae below $\mathcal{T}'$'s root that result from applications of inference rules.

Table 1: Transformation parameters.

it is normal. Nor can it result from an application of XFQ, because normal proofs are XFQ-canonical. So, $\neg p$ should belong to Λ_1 and the application of RB in (3) would be redundant (see Definition 4.1 of Part I and Example 4.1 of Part I) against the hypothesis that $\mathcal{T}$ is quasi normal, which implies that it contains no redundant applications of RB. Thus, it is impossible that p is an atomic formula that does not occur in $\Gamma \cup \{A\}$. Therefore, all applications of RB in $\mathcal{T}$ are analytic and $\mathcal{T}$ is normal. $\square$

Now, consider the parameter d_0 defined in (2) above, and recall the definitions of d_1–d_5 given in Part I that we recall here in Table 1 for the reader's convenience. Let

$$d(T) = \langle d_0(\mathcal{T}), d_1(\mathcal{T}), d_2(T), d_3(\mathcal{T}), d_4(\mathcal{T}), d_5(\mathcal{T})\rangle$$

and consider the usual lexicographic order on $d(\mathcal{T})$ for all C-intelim proofs $\mathcal{T}$. By inspection of the transformations (T1)–(T27) of Part I and of the transformations T28–T31 of this part, the reader can verify that each transformation that decreases $d_i(\mathcal{T})$ for any $i < 4$, may increase $d_j(\mathcal{T})$ for some $j > i$. However, no transformations that decreases $d_i(\mathcal{T})$ for $i > 0$, can ever increase d_j for any $j < i$. So, each of the transformations (T1)–(T31) decreases $d(\mathcal{T})$. Hence, the repeated application of these transformations, independently of their order, eventually yields a proof $\mathcal{T}'$ such that $d(\mathcal{T}') = \langle 0, 0, 0, 0, 0, 0\rangle$, which is therefore normal.

Theorem 1.3. *Any C-intelim proof of A depending on Γ can be transformed into a normal C-intelim proof of A depending on some $\Delta \subseteq \Gamma$ by means of any sufficiently long sequence of applications of the transformations* (T1)–(T31).

If we are interested in *atomically* normal proofs and not just in normal ones, all we need to do is change the definition of $g(\mathcal{T})$ in (1) as follows:

$$g(\mathcal{T}) = \begin{cases} \sharp(A) & \text{if } \mathcal{T} \text{ ends with a non-atomic application of RB and } A \text{ is the} \\ & \text{RB-formula of this application} \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Then it is not difficult to adapt the previous arguments to show:

Theorem 1.4. *Any C-intelim proof of A depending on Γ can be transformed into an atomically normal C-intelim proof of A depending on some $\Delta \subseteq \Gamma$ by means of any sufficiently long sequence of applications of the transformations* (T1)–(T31).

Restricting to normal proofs has several advantages over restricting to atomically normal proofs. Not only are normal proofs shorter in general, but they also allow for a notion of k-depth normal deducibility from a set of formulae (A is *k-depth normally deducible from* Γ if there is a k-depth normal proof of A depending on $\Delta \subseteq \Gamma$) that is *structural*, i.e., it is invariant under uniform substitutions of atomic formulae with arbitrary ones.

Indeed, we suggest that the *depth of a normal proof* provides a natural measure of the "difficulty" of the reasoning process represented by the proof. It reflects the maximum number of nested uses of "virtual information" (i.e., the assumptions discharged by an application of RB) in a proof. From the computational viewpoint, this idea is confirmed by the fact that, as we shall see Section 5, k-depth normal deducibility is a *tractable* problem.

2 The contamination problem

Let us say that two formulae are *syntactically disjoint* if they share no atomic formula. Two *sets* Γ and Δ are *syntactically disjoint* if every formula of Γ is syntactically disjoint from every formula of Δ. In what follows we shall write "$\Gamma \parallel \Delta$" for "Γ is syntactically disjoint from Δ".

It is routine to show that the following holds by classical semantics:

Theorem 2.1. *For every Γ and Δ, if*

1. $\Delta \parallel \Gamma \cup \{A\}$*, and*
2. $\Gamma \cup \Delta \vdash_{\mathsf{C}} A$*,*

then at least one of the following holds true:

- Γ *is consistent and* $\Gamma \vdash_{\mathsf{C}} A$*,*
- Γ *is inconsistent and* $\Gamma \vdash_{\mathsf{C}} A$*;*
- Δ *is inconsistent and* $\Delta \vdash_{\mathsf{C}} A$*.*

This holds for every (possibly empty) Γ, Δ. The special case in which Δ is empty is trivial.

In the special case in which Γ is empty, the theorem implies that:

Corollary 2.1. *If $\Delta \parallel \{A\}$, then $\Delta \vdash_{\mathsf{C}} A$ if and only if either A is a tautology or Δ is inconsistent.*

The *contamination problem* is the problem that arises in classical natural deduction when:

- for some *non-empty* Δ such that $\Delta \parallel \Gamma \cup \{A\}$, with $A \neq \curlywedge$, we have a natural deduction proof of A *depending*[3] on $\Gamma \cup \Delta$, or
- for some *non-empty* Δ, Γ such that $\Delta \parallel \Gamma$, we have a natural deduction proof of $\curlywedge$ depending on $\Gamma \cup \Delta$.

In the first case, it may be that $\Gamma \nvdash_{\mathsf{C}} A$, in which case, by classical semantics, Δ is inconsistent and assumptions that are totally unrelated either to the conclusion or to Γ play an active role in the proof, via a *practically meaningless* use of *ex-falso quodlibet*, to obtain a conclusion that could not have been otherwise obtained from Γ. Or it may be that $\Gamma \vdash_{\mathsf{C}} A$, in which case the assumptions in Δ, besides being unrelated, are also clearly unnecessary to obtain the conclusion. In the second case, either Δ or Γ must be inconsistent on their own and, again, unrelated and redundant assumptions are used to obtain the proof of $\curlywedge$. In any case such a proof violates a basic relevance condition that can indeed be satisfied by a better-behaved natural deduction proof, except for one distinguished case in which all of the following hold true:

1. Γ is empty,
2. Δ is inconsistent and cannot be be partitioned into two syntactically disjoint subsets,
3. the conclusion A is not equal to $\curlywedge$ and is obtained from Δ by means of an *ex-falso* (or better *ex-contradictione*) inference.

The simplest and paradigmatic example is the inference $p, \neg p/q$ displayed in (1), Part I. While such inferences cannot be expunged from any *complete* system for classical logic, we can easily obtain from them proof that Δ is inconsistent, that is a proof of $\curlywedge$ depending on Δ. In view of various applications of natural deduction (see the concluding section on this point) it would be useful to come up with a notion of normal proof that avoids incurring the contamination problem whenever possible, i.e. with the only exception of inferences satisfying the three conditions specified above, in which case, however, a refutation can be promptly obtained.

Definition 2.1 (Contaminated proofs)**.** *Given a natural deduction system S, we say that an S-proof π of A depending on Γ is* contaminated *if one of the following two conditions hold:*

[3] By this we mean that all the assumptions are actually used in the deduction tree.

$$\dfrac{A \to B \qquad \dfrac{\dfrac{C \quad \neg C}{\curlywedge}}{A}}{B} \qquad\qquad \dfrac{\dfrac{\dfrac{C \quad \neg C}{\curlywedge}}{B} \qquad \dfrac{\dfrac{\dfrac{D \quad \neg D}{\curlywedge}}{E} \qquad E \to \neg B}{\neg B}}{\curlywedge}$$

Figure 1: R-contaminated normal proofs in Prawitz's natural deduction.

1. *$A \neq \curlywedge$ and for some non-empty $\Delta \subseteq \Gamma$, $\Delta \parallel (\Gamma \setminus \Delta) \cup \{A\}$;*
2. *$A = \curlywedge$ and for some non-empty $\Delta \subset \Gamma$, $\Delta \parallel (\Gamma \setminus \Delta)$.*

In both cases we call Δ a contaminating set *for π.*

Definition 2.2 (R-contaminated proofs)**.** *We say that π is* redundantly contaminated *(*R-contaminated *for short) if (i) π is contaminated and (ii) there is a contaminating set Δ for π such that either $\Delta \subset \Gamma$ or $\Delta \nvdash \curlywedge$.*

Note that our definition implies that any contaminated *refutation* of Γ (a proof of $\curlywedge$ depending on Γ) is always R-contaminated. Proofs that are contaminated but not R-contaminated correspond to the exceptions discussed above that cannot be expunged by any natural deduction system that is complete for classical logic.[4]

Can we suitably restrict the notion of an acceptable natural deduction proof so as to deliver only proofs that are not R-contaminated? The notion of normal proof put forward in [Prawitz, 1965] is not sufficient for this purpose. The trees shown in Figure 1 represent R-contaminated proofs that are normal in Prawitz's sense (assuming that B and E are atomic to satisfy the restriction on the $\curlywedge_{\mathsf{C}}$ rule, see [Prawitz, 1965, Chapter III]). [5] On the other hand, we shall show, in the next section, that normal C-intelim proofs are never R-contaminated. Let us now focus on the "exceptional" proofs that are contaminated but not R-contaminated. These are proofs of A depending on a non-empty $\Gamma \parallel \{A\}$ in which $A \neq \curlywedge$ and Γ cannot be partitioned into syntactically disjoint subsets. We shall show later on that for a *normal* proof $\mathcal{T}$ this situation obtains only when $\mathcal{T}$ "contains", in a well-defined sense, a proof of $\curlywedge$ depending on the same set Γ of assumptions that can be *easily* extracted from it. We have already commented in the introduction (in Part I) that while such proofs cannot be excluded from a complete system for classical logic, since they express classically valid inferences, they are devoid of any practical value and that their *only* epistemic significance consists in revealing that the assumptions are inconsistent and therefore, from a classical logic perspective, cannot be used to make any reasonable inference at all.[6] This seems a good reason to call such proofs *improper*.

[4] They can be expunged if we switch to a paraconsistent system, such as the one that results from restricting to the *proper* normal proofs introduced in Definition 2.3 or from a classical extension of the system discussed in Tennant [1987].

[5] Note that neither can such proofs be turned into proofs of $\curlywedge$ depending on the *same* set of assumptions.

[6] This view is shared, of course, by all the advocates of relevance logic. However, with the notable exceptions of Timothy Smiley and Neil Tennant, most of them argue that disjunctive syllogism should be rejected as well as the ex-falso rule, given the role played by the latter in the proof of the former within the

Definition 2.3. *The notion of proper normal proof is defined inductively as follows:*

- *A normal* 0*-depth proof* $\mathcal{T}$ *is* proper *if its last step is not an application of XFQ (so* $\mathcal{T}$ *contains no applications of XFQ);*
- *for* $k > 0$*, a normal* k*-depth proof*

$$\mathcal{T} = \cfrac{\begin{matrix} [B] & [\neg B] \\ \mathcal{T}_1 & \mathcal{T}_2 \\ A & A \end{matrix}}{A}$$

is proper *if at least one of its immediate subproofs* $\mathcal{T}_1$ *and* $\mathcal{T}_2$ *is proper.*

It follows from the above definition that a normal C-intelim proof $\mathcal{T}$ is *proper* if at least one of its 0-depth components contains no application of XFQ. Observe that every normal refutation of Γ (i.e., a proof of $\curlywedge$ depending on Γ) is proper. This follows from the fact that normal proofs are regular, and therefore XFQ-canonical (Definitions 4.4 and 4.6 of Part I). Thus, XFQ can be applied only as the last step of one of its 0-depth components.

Let the *height* of a formula-tree be the maximal length of its branches.

Lemma 2.1. *If* $\mathcal{T}$ *is an improper normal proof of* A *depending on* Γ*, then* $\mathcal{T}$ *can be transformed into a normal refutation* $\mathcal{T}'$ *of* Γ *such that:*

1. $\mathsf{depth}(\mathcal{T}') = \mathsf{depth}(\mathcal{T})$,
2. $h(\mathcal{T}') = h(\mathcal{T}) - 1$.

Moreover, the computational cost of the transformation is linear in the number of applications of XFQ in $\mathcal{T}$*.*

Proof. If $\mathcal{T}$ improper, then all its 0-depth components (see Definition 4.8 of Part I) end with an application of XFQ. Hence, each 0-depth component contains, as a strict subproof, a normal proof of $\curlywedge$ depending on the same assumptions. Then all the virtual assumptions occurring in the 0-depth components can be discharged by applying RB with $\curlywedge$, instead of A, as conclusion. ☐

Fig. 2 shows an example of an improper normal proof with its associated normal refutation. Note that a proper normal proof does not need to depend on a consistent set of premises. A trivial example of a proper normal proof depending on inconsistent premises is the following:

$$\frac{A \quad \neg A}{A \wedge \neg A} \tag{5}$$

framework of Gentzen-style natural deduction. By contrast, disjunctive syllogism is a primitive rule in the C-intelim system.

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to B \quad [A]^1}{B} \quad \dfrac{A \to \neg B \quad [A]^1}{\neg B}}{⅄}}{D} \qquad \dfrac{\dfrac{\dfrac{\neg A \to C \quad [\neg A]^2}{C} \quad \dfrac{\neg A \to \neg C \quad [\neg A]^2}{\neg C}}{⅄}}{D}}{D}\;{}_{1,2}$$

$$\dfrac{\dfrac{\dfrac{A \to B \quad [A]^1}{B} \quad \dfrac{A \to \neg B \quad [A]^1}{\neg B}}{⅄} \qquad \dfrac{\dfrac{\neg A \to C \quad [\neg A]^2}{C} \quad \dfrac{\neg A \to \neg C \quad [\neg A]^2}{\neg C}}{⅄}}{⅄}\;{}_{1,2}$$

Figure 2: Transforming an improper normal proof of D into a normal refutation.

3 Variable sharing and non-contamination

We shall now show that normal C-intelim proofs are never R-contaminated and that *proper* normal proofs are never contaminated, so that they enjoy the variable-sharing property, except for the case in which their conclusion is $⅄$. We start with a lemma on 0-depth proofs.

Lemma 3.1. *For every non-empty* Γ *and every* A*, there is no* 0*-depth proper normal proof of* A *depending on* Γ *such that* $A \neq ⅄$ *and* $\Gamma \parallel \{A\}$.

Proof. Let $\mathcal{T}$ be a 0-depth proper normal proof of A depending on Γ. Being proper, $\mathcal{T}$ contains no applications of XFQ. Moreover, if $A \neq ⅄$, it also contains no application of RNC. By Lemma 1.1, at least one branch of $\mathcal{T}$ is a complete normal intelim walk. So, there are four cases: (i) the E-part and the I-part are both empty; then the minimum formula is the only element of the walk and the whole proof $\mathcal{T}$ is a trivial one-node proof of A depending on A; (ii) the E-part is non-empty and the I-part is empty, in which case the minimum formula is equal to the conclusion A and a proper weak subformula of some assumption in Γ; (iii) the E-part is empty and the I-part is non-empty, in which case the minimum formula is at the same time one of the assumptions in Γ and a proper weak subformula of the conclusion A; (iv) neither the E-part nor the I-part are empty, in which case the minimum formula is a proper weak subformula both of some assumption in Γ and of the conclusion A. In all cases the minimum formula of the complete intelim walk is at the same time a weak subformula of some formula in Γ and of A. Hence, $\Gamma \nparallel \{A\}$. □

It follows that in a normal 0-depth proof $\mathcal{T}$ of A depending on a non-empty Γ, if $\Gamma \parallel \{A\}$, either $A = ⅄$ or $\mathcal{T}$ is improper, that is, the proof ends with an application of XFQ. In either case Γ is inconsistent.

Corollary 3.1. *For every non-empty* Γ *and every* A*, if there is a normal* 0*-depth proof of* A *depending on* Γ *such that* $\Gamma \parallel \{A\}$*, then* Γ *is inconsistent.*

Notice that there are no 0-depth proofs of a tautology depending on the empty set of assumptions. Despite this, 0-depth deducibility is a Tarskian consequence relation. The same absence of tautologies is shared by other logics based on informational notions, such as Kleene's 3-valued and Belnap's 4-valued logics. Interestingly, the 0-depth logic, like intuitionistic logic, admits of no finitely-valued characteristic matrix.[7] However, it admits of an intuitive semantics based on a non-deterministic matrix that (to the best of our knowledge) was first discussed by W.V.O. Quine in order to fix what he called the "primitive meaning of the logical operators" [Quine, 1973, §20].[8]

We are now in a position to show that the variable sharing property applies to proper normal proofs of arbitrary depth.

Theorem 3.1 (Variable-sharing property)**.** *If $\mathcal{T}$ is a proper normal proof of $A \neq \curlywedge$ depending on Γ, then $\Gamma \not\parallel \{A\}$.*

Proof. Observe that, every 0-depth component of $\mathcal{T}$ is a 0-depth proper normal proof of A depending on $\Gamma' \cup \Delta$ such that $\Gamma' \subseteq \Gamma$ and Δ is the set of virtual assumptions introduced in this 0-depth component that are subsequently discharged. By Lemma 3.1, each 0-depth component has the variable sharing property. Since$\mathcal{T}$ is normal, all the formulae in Δ are weak subformulae of some formula in $\Gamma \cup \{A\}$. □

Although proper normal proofs are not complete for classical logic, they are complete for the set of valid inferences from consistent sets of assumptions, since if there is no proper normal proof of A depending on Γ, then Γ must be inconsistent, in that any improper proof, by Lemma 2.1, "contains" a refutation of Γ. As shown above, they also enjoy the variable sharing property. Indeed, the system of deduction that accepts as admissible only proper normal proofs has close connections with Tennant-style relevance logic [Tennant, 1984, 1987]. However, it is hard to take it, as it stands, as a well-behaved system of relevance logic. For example, there is a proper normal proof of $(A \vee B) \wedge B$ depending on A and $\neg A$, which is only one breadth away from $A, \neg A/B$. It must be noticed, however, that there is no proper normal proof that is also atomically normal. The connection between our approach and Tennant-style relevance logic will be investigated in a future paper.

Theorem 3.2 (Weak non-contamination)**.** *if $\mathcal{T}$ is a normal proof of A depending on Γ, then $\mathcal{T}$ is not R-contaminated.*

Proof. First, note that if there is a contaminating Δ such that the first disjunct of condition (ii) in Definition 2.2 is false, that is, Γ itself is contaminating for $\mathcal{T}$, then the other disjunct must also be false, for in such a case $\Delta = \Gamma \parallel \{A\}$ and, by Theorem 3.1, $\mathcal{T}$ is improper. Hence, every 0-depth component of $\mathcal{T}$ ends with an application of XFQ. Such an improper proof can be easily turned into a refutation of $\Gamma = \Delta$ (see Lemma 2.1) and so $\Delta \vdash \curlywedge$. Hence, to show the theorem it is sufficient to show that there is no contaminating Δ properly included in Γ.

To spare on parentheses, we shall assume throughout this proof that "$\setminus$" binds more tightly than "$\cup$". The proof is by induction on the height of $\mathcal{T}$ (the maximal length of a branch of $\mathcal{T}$) denoted by $h(\mathcal{T})$.

[7] See [D'Agostino et al., 2013] where this result is proven for a different, tableau-like, variant of C-intelim.

[8] See [D'Agostino, 2014, 2015] for an in-depth discussion including soundness and completeness results.

Base: $h(\mathcal{T}) = 1$. Then $\mathcal{T}$ is a one-node formula tree representing a proper normal proof of A depending on A. Then, trivially, $\mathcal{T}$ is non-contaminated.

Step: $h(\mathcal{T}) = k > 1$. The theorem holds for every normal proof $\mathcal{T}'$ such that $h(\mathcal{T}') < k$. We show that it holds also for $\mathcal{T}$. There are several cases depending on the last step of $\mathcal{T}$.

Case 1: the last step of $\mathcal{T}$ is an application of XFQ. Then, since in a normal proof XFQ can be applied only as the last step in a 0-depth component, $\mathsf{depth}(\mathcal{T}) = 0$, $\mathcal{T}$ is improper and has the following form:

$$\frac{\begin{array}{c}\mathcal{T}_1\\ \curlywedge\end{array}}{A.}$$

By inductive hypothesis $\mathcal{T}_1$ is not R-contaminated and so, by Definition 2.1, there is no $\Delta \subset \Gamma$ such that $\Delta \parallel \Gamma \setminus \Delta$. Hence, there is no contaminating Δ for $\mathcal{T}$ properly included in Γ and $\mathcal{T}$ is not R-contaminated.

Case 2: the last step of $\mathcal{T}$ is an application of an introduction rule, then $\mathsf{depth}(\mathcal{T}) = 0$. We discuss only the sub-cases in which the introduction rule is one of those involving $\vee$, the others being similar. So, $\mathcal{T}$ has one of the following forms:

$$\frac{\begin{array}{c}\mathcal{T}_1\\ A\end{array}}{A \vee B} \qquad \frac{\begin{array}{c}\mathcal{T}_1\\ B\end{array}}{A \vee B} \qquad \frac{\begin{array}{c}\mathcal{T}_1\\ \neg A\end{array} \quad \begin{array}{c}\mathcal{T}_2\\ \neg B\end{array}}{\neg(A \vee B)}$$

For the first two sub-cases, let Γ be the set of all the assumptions of $\mathcal{T}$. By inductive hypothesis, $\mathcal{T}_1$ is not R-contaminated. Suppose, *ex-absurdo*, that $\mathcal{T}$ is R-contaminated, i.e., there is a non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta \cup \{A \vee B\}$$

Then

$$\Delta \parallel \Gamma \setminus \Delta \cup \{A\} \text{ and } \Delta \parallel \Gamma \setminus \Delta \cup \{B\},$$

and so in both cases $\mathcal{T}_1$ would be R-contaminated against the inductive hypothesis.

As for the third sub-case, let Γ_1 and Γ_2 be, respectively, the sets of all the assumptions of $\mathcal{T}_1$ and $\mathcal{T}_2$. By inductive hypothesis, neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Moreover, they are both proper, given that $\mathcal{T}$ is. Suppose now, *ex-absurdo* that $\mathcal{T}$ is R-contaminated, i.e., $\Delta \parallel \Gamma \setminus \Delta \cup \{\neg(A \vee B)\}$ for some non-empty $\Delta \subset \Gamma$. Let $\Delta_1 = \Gamma_1 \cap \Delta$ and $\Delta_2 = \Gamma_2 \cap \Delta$. Then:

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{\neg A\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{\neg B\}.$$

Since Δ is a non-empty proper subset of Γ, then either Δ_1 is a non-empty proper subset of Γ_1 or Δ_2 is a non empty proper subset of Γ_2. To see this, first observe that, since $\mathcal{T}_1$ and $\mathcal{T}_2$ are both 0-depth proper normal proofs and their conclusions are different from $\curlywedge$, neither $\Delta_1 = \Gamma_1$, nor $\Delta_2 = \Gamma_2$, otherwise $\Gamma_1 \parallel \{\neg A\}$ or $\Gamma_2 \parallel \{\neg B\}$, against Lemma 3.1. Moreover, suppose Δ_1 (Δ_2) is empty. Then, since $\Delta \subset \Gamma$, Δ_2 (Δ_1)

cannot be empty and must be a proper subset of Γ_2 (Γ_1). Therefore, at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated against the inductive hypothesis.

Case 3: the last step of $\mathcal{T}$ is an application of an elimination rule, then *depth*$(\mathcal{T}) = 0$. We discuss only the sub-cases in which the elimination rule is one of those involving $\vee$, the others being similar. So, $\mathcal{T}$ has one of the following forms:

$$\frac{\begin{matrix}\mathcal{T}_1 & \mathcal{T}_2\\ A \vee B & \neg A\end{matrix}}{B} \qquad \frac{\begin{matrix}\mathcal{T}_1 & \mathcal{T}_2\\ A \vee B & \neg B\end{matrix}}{A} \qquad \frac{\begin{matrix}\mathcal{T}_1\\ \neg(A \vee B)\end{matrix}}{\neg A} \qquad \frac{\begin{matrix}\mathcal{T}_1\\ \neg(A \vee B)\end{matrix}}{\neg B}$$

Consider the first sub-case. By inductive hypothesis neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Moreover, they are both proper, since $\mathcal{T}$ is. Suppose *ex absurdo* that $\mathcal{T}$ is R-contaminated, i.e, there is some non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta \cup \{B\}.$$

Let Γ_1 and Γ_2 be, respectively, the assumptions of $\mathcal{T}_1$ and $\mathcal{T}_2$. Let also $\Delta_1 = \Gamma_1 \cap \Delta$ and $\Delta_2 = \Gamma_2 \cap \Delta$. Then

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{B\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{B\}.$$

By Corollary 1.1, $A \vee B$ is a subformula of some formula $C \in \Gamma_1 \cup \{B\}$. Clearly it is not a subformula of B; moreover C cannot occur in Δ, otherwise Δ would not be syntactically disjoint from $\{B\}$. Hence, C occurs in $\Gamma \setminus \Delta$ and $\Delta \parallel \{C\}$; so both $\Delta_1 \parallel \{C\}$ and $\Delta_2 \parallel \{C\}$. Given that $A \vee B$ is a subformula of C, $\Delta_1 \parallel \{A \vee B\}$ and $\Delta_2 \parallel \{\neg A\}$. Then,

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{A \vee B\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{\neg A\}.$$

Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are both proper and their conclusions are different from $\curlywedge$, neither $\Delta_1 = \Gamma_1$, nor $\Delta_2 = \Gamma_2$, otherwise $\Gamma_1 \parallel \{\neg A \vee B\}$ or $\Gamma_2 \parallel \{\neg A\}$, against Lemma 3.1. As in the previous case, it can be easily shown that, either Δ_1 is a non-empty proper subset of Γ_1 or Δ_2 is a non-empty proper subset of Γ_2, and so at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated, against the inductive hypothesis. The proof is similar for the second sub-case.

As for the third sub-case, suppose there is a non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta \cup \{\neg A\}.$$

By Corollary 1.1, $\neg(A \vee B)$ is a weak subformula of some formula $C \in \Gamma \cup \{\neg A\}$. Clearly it is not a weak subformula of $\neg A$. Again, C cannot occur in Δ, otherwise Δ would not be syntactically disjoint from $\{\neg A\}$. Thus C occurs in $\Gamma \setminus \Delta$. Since $\Delta \parallel \Gamma \setminus \Delta$, then $\Delta \parallel \{C\}$ and, therefore, $\Delta \parallel \{\neg(A \vee B)\}$. Hence, $\Delta \parallel \Gamma \setminus \Delta \cup \{\neg(A \vee B)\}$ and so $\mathcal{T}_1$ is R-contaminated, against the inductive hypothesis. The fourth sub-case is similar to the third.

Case 4: the last step of $\mathcal{T}$ is an application of RNC. Then $\mathsf{depth}(\mathcal{T}) = 0$ and $\mathcal{T}$ has the following form:

$$\frac{\begin{matrix}\mathcal{T}_1 & \mathcal{T}_2\\ B & \neg B\end{matrix}}{\curlywedge}$$

where, by inductive hypothesis, neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Moreover, they are both proper, given that $\mathcal{T}$ is, and $B \neq \curlywedge$. Suppose, *ex absurdo*, that $\mathcal{T}$ is R-contaminated. This means that there is a non-empty $\Delta \subset \Gamma$ such that

$$\Delta \parallel \Gamma \setminus \Delta.$$

Let Γ_1 and Γ_2 be, respectively, the assumptions of $\mathcal{T}_1$ and $\mathcal{T}_2$. Let also $\Delta_1 = \Gamma_1 \cap \Delta$ and $\Delta_2 = \Gamma_2 \cap \Delta$. Then,

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2.$$

By Theorem 1.1, either $B \in \Gamma$ or B is a proper weak subformula of some formula $C \in \Gamma$. In either case, B is a weak subformula of some $C \in \Gamma$. Now, either $C \in \Delta$ or $C \in \Gamma \setminus \Delta$.

In the first case, it follows that $\{B\} \parallel \Gamma \setminus \Delta$. Therefore:

$$\{B\} \parallel \Gamma_1 \setminus \Delta_1 \text{ and } \{\neg B\} \parallel \Gamma_2 \setminus \Delta_2.$$

Therefore:

$$\Gamma_1 \setminus \Delta_1 \parallel \Delta_1 \cup \{B\} \text{ and } \Gamma_2 \setminus \Delta_2 \parallel \Delta_2 \cup \{\neg B\},$$

Observing that $\Delta_1 = \Gamma_1 \setminus (\Gamma_1 \setminus \Delta_1)$ and $\Delta_2 = \Gamma_2 \setminus (\Gamma_2 \setminus \Delta_2)$,

$$\Gamma_1 \setminus \Delta_1 \parallel \Gamma_1 \setminus (\Gamma_1 \setminus \Delta_1) \cup \{B\} \text{ and } \Gamma_2 \setminus \Delta_2 \parallel \Gamma_2 \setminus (\Gamma_2 \setminus \Delta_2) \cup \{\neg B\},$$

As in the previous cases, given that $\mathcal{T}_1$ and $\mathcal{T}_2$ are both 0-depth proper normal proofs, it can be easily shown, using Lemma 3.1, that either $\Gamma_1 \setminus \Delta_1$ is a non-empty proper subset of Γ_1 or $\Gamma_2 \setminus \Delta_2$ is a non-empty proper subset of Γ_2, and so at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated, against the inductive hypothesis.

In the second case, it follows that $\{B\} \parallel \Delta$. Therefore:

$$\{B\} \parallel \Delta_1 \text{ and } \{\neg B\} \parallel \Delta_2.$$

Hence:

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{B\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{\neg B\}.$$

Again, this implies that at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is either improper (by Lemma 3.1) or R-contaminated, against the inductive hypothesis.

Case 5: The last step of $\mathcal{T}$ is an application or RB. Then $\mathcal{T}$ is a k-depth proof of A depending on Γ with $k > 0$ and has the following form:

$$\frac{\begin{array}{cc} [B] & [\neg B] \\ \mathcal{T}_1 & \mathcal{T}_2 \\ A & A \end{array}}{A}$$

By inductive hypothesis neither $\mathcal{T}_1$ nor $\mathcal{T}_2$ are R-contaminated. Let $\Gamma_1, \Gamma_2, \Delta_1, \Delta_2$ be defined as usual. Suppose that $\mathcal{T}$ is R-contaminated, that is, for some non-empty $\Delta \subset \Gamma$ we have that:

$$\Delta \parallel \Gamma \setminus \Delta \cup \{A\}, \tag{6}$$

and therefore:

$$\Delta_1 \parallel \Gamma_1 \setminus \Delta_1 \cup \{A\} \text{ and } \Delta_2 \parallel \Gamma_2 \setminus \Delta_2 \cup \{A\}. \tag{7}$$

Since both $\mathcal{T}_1$ and $\mathcal{T}_2$ are normal, and so all the applications of RB are analytic, B is a subformula of some $C \in \Gamma \cup \{A\}$.

Now, either $C \in \Delta$ or $C \in \Gamma \setminus \Delta \cup \{A\}$. In the first case, by (6),

$$\{B\} \parallel \Gamma_1 \setminus \Delta_1 \cup \{A\} \text{ and } \{\neg B\} \parallel \Gamma_2 \setminus \Delta_2 \cup \{A\}.$$

It follows that:

$$\Delta_1 \cup \{B\} \parallel (\Gamma_1 \cup \{B\}) \setminus (\Delta_1 \cup \{B\}) \cup \{A\} \tag{8}$$

and

$$\Delta_2 \cup \{\neg B\} \parallel (\Gamma_2 \cup \{\neg B\}) \setminus (\Delta_2 \cup \{\neg B\}) \cup \{A\}. \tag{9}$$

Observe that both $\Delta_1 \cup \{B\}$ and $\Delta_2 \cup \{\neg B\}$ are non-empty. Moreover, since Δ is a *proper* subset of Γ it cannot be the case that both $\Delta_1 = \Gamma_1$ and $\Delta_2 = \Gamma_2$, otherwise it should be $\Delta = \Gamma$. Thus, at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated against the inductive hypothesis.

In the second case, it follows from (6) that $\Delta \parallel \{B\}$, and so:

$$\Delta_1 \parallel \{B\} \text{ and } \Delta_2 \parallel \{\neg B\}. \tag{10}$$

Therefore, by (7),

$$\Delta_1 \parallel (\Gamma_1 \cup \{B\}) \setminus \Delta_1 \cup \{A\} \text{ and } \Delta_2 \parallel (\Gamma_2 \cup \{\neg B\}) \setminus \Delta_2 \cup \{A\}. \tag{11}$$

Now, at least one of Δ_1 and Δ_2 is non-empty. Moreover Δ_1 (Δ_2) is by definition a subset of Γ_1 (Γ_2) and, by (10), does not contain B ($\neg B$). Therefore, either Δ_1 is a non-empty proper subset of $(\Gamma_1 \cup \{B\})$ or Δ_2 is a non-empty proper subset of $(\Gamma_2 \cup \{\neg B\})$. Hence, at least one of $\mathcal{T}_1$ and $\mathcal{T}_2$ is R-contaminated, against the inductive hypothesis. □

Putting together Lemma 2.1, Theorem 3.1 and Theorem 3.2 we have shown that:

Corollary 3.2 (Non-Contamination Property)**.** *If $\mathcal{T}$ is a normal proof of A depending on Γ, either $\mathcal{T}$ is non-contaminated or $\mathcal{T}$ is improper and can be turned in linear time into a non contaminated proof of $\curlywedge$ depending on Γ.*

4 C-intelim tableaux

The format of C-intelim proofs that we have presented so far, where proofs are trees with the conclusion as root and the assumptions as leaves, allows for easy comparison with Gentzen-style natural deduction, which exhibits the same formal structure. Although very perspicuous, this format involves a good deal of redundancy in the representation of proofs. Whenever a formula, which can be inferred from the assumptions, is used more than once as a premise of further inferences, its proof tree has to be

1	$(A \vee B) \to \neg(C \vee D)$	Assumption
2	A	Assumption
3	$(A \wedge E) \to C$	Assumption
4	$A \vee B$	$\vee\mathcal{I}1$ (2)
5	$\neg(C \vee D)$	$\to\mathcal{E}1$ (1,4)
6	$\neg C$	$\neg\vee\mathcal{E}1$ (5)
7	$\neg D$	$\neg\vee\mathcal{E}2$ (5)
8	$\neg(A \wedge E)$	$\to\mathcal{E}2$ (3,6)
9	$\neg E$	$\neg\wedge\mathcal{E}1$ (8,2).

Figure 3: A C-intelim sequence.

replicated, as in Figure 1 of Part I, where the derivation of $\neg B$ from A and $A \to \neg B$ is repeated twice. We shall therefore shift to a different format, that we call here *C-intelim tableaux*, that provides a more concise representation of arguments and an easier implementation of the RB-rule, bringing the format of C-intelim proofs closer to that of Smullyan's [Smullyan, 1968] or KE tableaux [D'Agostino and Mondadori, 1994], with the notable difference that C-intelim tableaux can represent direct proofs as well as refutations. A C-intelim tableau can be easily turned into a C-intelim natural deduction proof in the standard format. Readers who find the latter more perspicuous as a presentation of proofs, could look at the content of this section as a step towards the development of efficient *automated proof search procedures* for the natural deduction system of the previous sections. On the other hand, readers who are more familiar with Smullyan's tableaux and KE may look at C-intelim tableaux as an extension of KE that includes also introduction rules and allows for direct proofs as well as for refutations.

In this new format the application of the intelim rules is *sequential*: their premises do not occur on adjacent branches, but on the same branch as the conclusion and anywhere above it. Moreover, as will be seen below, there is no need for explicit falsum rules such as RNC and XFQ. As a result, 0-depth proofs, i.e., the ones involving no application of the proof rule RB, can be represented as *intelim sequences*.

Definitions 4.1 (C-intelim sequence)**.** *A* C-intelim sequence *is a sequence of formulae such that each formula is either (i) an assumption, or (ii) the conclusion of the application of an intelim rule to preceding formulae.*

A C-intelim sequence based on Γ *is a C-intelim sequence such that all its assumptions belong to* Γ.

A C-intelim sequence is closed *if it contains both A and $\neg A$ for some formula A, otherwise it is* open.

The array of formulae in Fig. 3 is a C-intelim sequence based on the set $\{(A \vee B) \to \neg(C \vee D), A, (A \wedge E) \to C\}$. The sequence starts by listing the assumptions and each subsequent formula is obtained by an application of an intelim rule to preceding formulae. In this example, for the reader's convenience, the full justification of each formula in the sequence is specified on its right. In the sequel the justification will be omitted and left to the reader. (Therefore numbering the formulae will no longer be required.) Note that the change of format implies that an intelim sequence may well contain *idle* occurrences of formulae, namely occurrences of formulae other than the last one that are not used as premises of a rule application, such as the one in step 7. This is not possible when 0-depth proofs are represented as formula-trees as in the previous sections, where all the formula occurrences in the tree, except for the end formula occurring at the root, are used as premises of some rule application. When a proof is completed, such idle formulae can be simply removed without affecting soundness. It may well be that removing an idle formula makes idle some of the premises used to obtain it, which can be, in turn, removed until no idle formula is left in the tree.

Observe also that each intelim sequence based on Γ can be easily turned into a 0-depth C-intelim proof of its last formula depending on some subset Δ of Γ, provided that all idle occurrences of formulae are first removed from the sequence.

In this new format, the Rule of Bivalence (RB) is a *branching rule* that splits an intelim sequence into two branches as follows (depending on whether it is used for signed or unsigned formulae):

$$\begin{array}{ccc} & \vdots & \\ & \diagup \quad \diagdown & \\ T\,A & & F\,A \end{array} \qquad\qquad \begin{array}{ccc} & \vdots & \\ & \diagup \quad \diagdown & \\ A & & \neg A \end{array}$$

So, deductions are again represented as trees, except that these trees now grow upside-down, like analytic tableaux. When a deduction tree is expanded in this way we say that RB has been applied to the formula A and that A is *the RB-formula* of this application of RB. Each application of this rule introduces, on each of the two branches, an extra assumption that we call *virtual assumption* to distinguish it from the *actual* assumptions displayed at the beginning, in such a way that exactly one of these two virtual assumptions must be true as a consequence of the classical Principle of Bivalence. This is the only branching rule of the system.

In the following definition we use the expression "tree of formulae" as an abbreviation of "tree whose nodes, except possibly the root, are labelled with formulae". (The special case when the root is unlabelled will be used to represent proofs from the empty set of assumptions, as will be explained below.)

Definition 4.1. *A* C-intelim tableau *is a tree of formulae such that each formula occurrence is either (i) an actual assumption, or (ii) results from previous formula occurrences in the same branch by an application of an intelim rule, or (iii) is one of the complementary virtual assumptions introduced by an application of the branching rule RB.*

A C-intelim tableau based on a set Γ *of formulae is a C-intelim tableau such that all its actual assumptions belong to* Γ.

Observe that each branch of a C-intelim tableau based on Γ is an intelim sequence based on $\Gamma \cup \Delta$ where Δ is the set of virtual assumptions introduced by the applications of RB in that branch. We say that a branch of a C-intelim tableau is *closed* when it contains both A and $\neg A$ for some formula A, otherwise we say that it is *open*. A C-intelim *tableau* is *closed* when all its branches are closed, otherwise it is *open*. Then the notion of *C-intelim tableau proof* can be defined as follows:

Definition 4.2. *A* C-intelim tableau proof *of A from Γ is an intelim tableau $\mathcal{T}$ based on Γ such that A occurs in each open branch of $\mathcal{T}$.*

Observe that this definition allows us to dispense with XFQ and implies that, if there are no open branches, $\mathcal{T}$ is a C-intelim proof of any formula A from Γ.

Definition 4.3. *A* C-intelim tableau refutation *of Γ is a closed C-intelim tableau based on Γ.*

Let us write $\Gamma \vdash^{IET} A$ if there is C-intelim tableau proof of A from Γ. Being essentially an alternative way of representing C-intelim proofs, C-intelim tableaux are complete for classical logic:

$$\text{If } A \text{ is a classical consequence of } \Gamma\text{, then } \Gamma \vdash^{IET} A.$$

Examples of C-intelim tableau proofs based on non-empty sets of assumptions are shown in Figure 4. The first one is a proof of G that contains a closed branch, and an open branch ending in G that correspond, respectively, to the left and right subproofs of the proof in Figure 1 in Part 1. Closed branches are marked with the symbol $\times$. It is customary to list all the actual assumptions at the beginning starting from the root. Proofs from the empty set of assumptions are represented by trees with an unlabelled root as illustrated in Figure 5. The first example shows how the introduction rules can be used to simulate the truth-table method. The second example shows how to represent a typical pattern of proof ex-absurdo. C-intelim tableaux can be naturally used as a *refutation system*, like resolution or semantic tableaux, as well as a system of direct proof. In fact, if we disallow the introduction rules, we obtain the system KE (Mondadori [1988a], D'Agostino and Mondadori [1994]), which is a variant of analytic tableaux (but essentially more efficient).[9] On the other hand if we disallow the elimination rules we obtain the system KI (Mondadori [1988b, 1995], D'Agostino [1999]), which can be regarded as a proof-theoretical version of the truth-table method (but essentially more efficient).[10] Using both introduction and elimination rules allows for shorter[11] and more natural deductions that require fewer applications of the proof rule RB.

Note that C-intelim tableaux directly correspond to proofs in the conventional format of the previous sections that are both RB and XFQ canonical, i.e., such that all the applications of RB and XFQ have been pushed downwards. A branch corresponds to a 0-depth component. In general a k-depth C-intelim proofs corresponds to a C-intelim tableau that contains at most k nested applications of the RB rule (i.e., such that the maximal number of virtual assumptions in a branch is k). However, the applications of RNC and XFQ are no longer necessary, since they are absorbed by the notion of closed branch and by the definition of C-intelim tableau proof. A *path* in a C-intelim tableau $\mathcal{T}$ is a finite sequence of nodes such that the first node is the root of $\mathcal{T}$ and each subsequent node occurs immediately below the previous one (so, a branch is a maximal

[9] As shown in D'Agostino and Mondadori [1994], KE can p-simulate analytic tableaux but analytic tableaux cannot p-simulate KE. In fact, analytic tableaux cannot even p-simulate the truth-tables (D'Agostino [1992]).

[10] The truth-table method cannot p-simulate KI (Mondadori [1995]).

[11] But not *essentially* shorter, for both KI and KE can p-simulate C-intelim (D'Agostino [1999]).

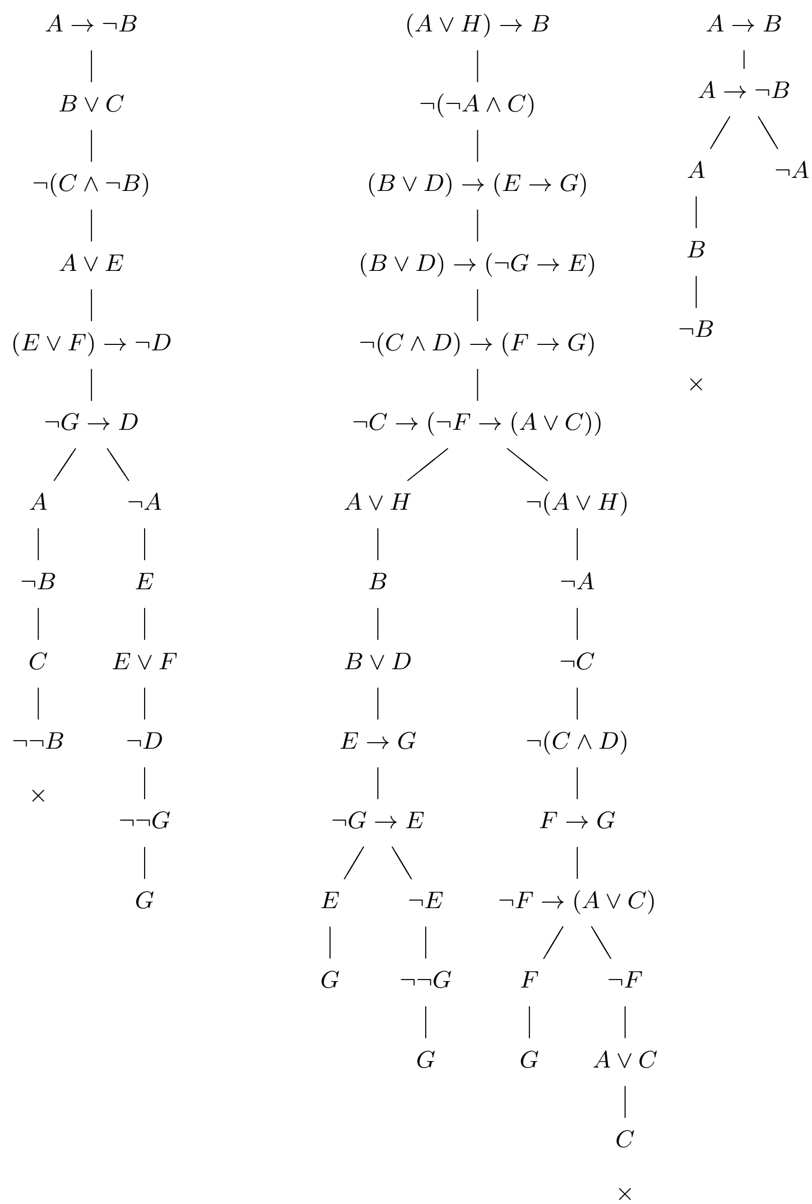

Figure 4: C-intelim tableaux. Each branch is an intelim sequence.

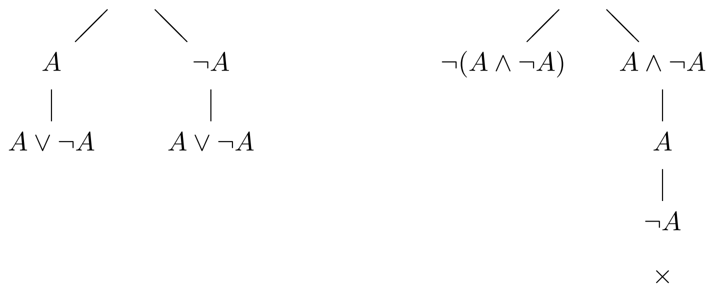

Figure 5: C-intelim tableau proofs from the empty set of assumptions.

path). A path is *closed* if it contains occurrence of both B and $\neg B$ for some formula B.

The notion of quasi-normal C-intelim proof is replaced by a much simpler (and somewhat stronger) notion of *non-redundant* C-intelim tableaux.

Definition 4.4. *A C-intelim tableau proof* $\mathcal{T}$ *of* A *from* Γ *(refutation of* Γ*) is* non-redundant *if the following conditions are satisfied:*

1. $\mathcal{T}$ *contains no idle occurrences of formulae;*
2. *no branch of* $\mathcal{T}$ *contains more than one occurrence of the same formula;*
3. *no branch of* $\mathcal{T}$ *properly contains a closed path.*

The reader can verify that the two conditions 2 and 3 are sufficient to ensure that a non-redundant C-intelim tableau contains no detours, where detours are defined as before (see Definition 4.5 of Part I) and that every non-redundant C-intelim tableau can be easily represented as a quasi-normal C-intelim proof, adding applications of RNC and, possibly, of XFQ at the end of each 0-depth component resulting from a closed branch. The notion of non-redundant C-intelim tableau, however, is stronger than the notion of quasi-normal C-intelim proof defined in Part I, in that it removes redundancies that are not purged by the notion of quasi-normal proof. For example, the following:

$$\frac{A \quad A \to A}{A} \tag{12}$$

is a quasi-normal C-intelim proof in the sense of Part I (since it contains no application of RB it is also trivially normal). However, its translation into a (one-branch) C-intelim tableau:

$$\begin{array}{c} A \\ A \to A \\ A \end{array} \tag{13}$$

is redundant because it violates condition 2 in Definition 4.4. The only non-redundant version of the above C-intelim proof is the trivial one containing only one occurrence of A. This example suggests that an obvious enhancement of the notion of quasi-normal proof in the conventional format would be requiring that the same formula never occurs more than once in the same branch. This enhancement would indeed rule out the redundant proof in (12), but would not be sufficient to rule out the following:

$$\frac{\dfrac{\dfrac{A \quad A \to B}{B} \qquad B \to C}{C} \qquad \dfrac{\dfrac{E \quad E \to B}{B} \qquad B \to D}{D}}{C \wedge D} \tag{14}$$

which is still redundant because the conclusion B is obtained twice from different sets of assumptions. On the other hand, the corresponding C-intelim tableau:

$$\begin{array}{c} A \\ A \to B \\ B \to C \\ E \\ E \to B \\ B \to D \\ B \\ C \\ D \\ C \wedge D \end{array} \tag{15}$$

is redundant, since it contains idle occurrences of formulae. These are either A and $A \to B$, or E and $E \to B$, depending on which pair is used in deriving the conclusion B. The C-intelim sequence becomes non-redundant if one of the two pairs of formulae is removed from it. It is indeed possible to further enhance the notion of quasi-normal C-intelim proof of Part I to make it correspond exactly to the notion of non-redundant C-intelim tableau, at the price of some complication. On the other hand, it seems remarkable that such a simple notion of non-redundancy as the one given in Definition 4.4 is sufficient not only to ensure that a C-intelim tableau complying with it contains no detours, but also that it does not contain other obvious redundancies.

The procedure to turn a C-intelim tableau into a non-redundant one is quite trivial:

1. if a branch properly contains a closed path, remove all the nodes following the closed path;

2. remove, one by one, all idle formulae and all repetitions of the same formula in a branch;

3. If the idle or repeated formula is a virtual assumption introduced by an application of RB, we must remove also the whole subtree below the sibling node.

The elimination of idle or repeated occurrences of a formula in a branch may turn some previously used occurrences of formulae into idle ones; but at each reduction step the size of the tree decreases, and so the procedure terminates in a number of steps that is linear in the size of the initial tableau.

Definition 4.5. *A C-intelim tableau proof or refutation is* normal *if it is non-redundant and all applications of RB in it are analytic. It is* atomically normal *if it is normal and all applications or RB are atomic.*[12]

Note that a *normal* C-intelim tableaux proof of A from Γ can be easily turned to a normal C-intelim proof of A depending on Γ which is essentially identical, except for the format and for the applications of RNC and, possibly, XFQ at the end of its 0-depth components (whenever the latter correspond to a closed branch). However, the opposite is not true because some normal C-intelim proofs may fail to fully satisfy the non-redundancy condition.

The proofs of Theorem 1.2, Theorem 1.3 and Theorem 3.2 can be easily adapted to yield the following:

Theorem 4.1. *If $\mathcal{T}$ is a normal C-intelim tableaux, then $\mathcal{T}$ has the WSFP.*

Theorem 4.2. *Every C-intelim tableau proof of A from Γ (refutation of Γ) can be transformed into a(n atomically) normal one of A from some $\Delta \subseteq \Gamma$.*

Theorem 4.3. *If $\mathcal{T}$ is a normal C-intelim tableaux, then $\mathcal{T}$ is not R-contaminated.*

The *depth* of a C-intelim tableau $\mathcal{T}$ is simply the maximum number of virtual assumptions occurring in one of its branches.

Definition 4.6. *$\mathcal{T}$ is a* proper *normal C-intelim tableau proof of A based on Γ if $\mathcal{T}$ is normal C-intelim tableau proof of A based on Γ and $\mathcal{T}$ is open (at least one of its branches is not closed).*

Again, the proof of the next theorems is parasitic on the proof of their analogues for C-intelim proofs (namely, Theorem 3.1 and Corollary 3.2).

Theorem 4.4. *If $\mathcal{T}$ is a proper normal C-intelim tableau proof of A from Γ and $A \neq \curlywedge$, then $\Gamma \nVdash \{A\}$.*

Theorem 4.5 (Non-contamination property)**.** *If $\mathcal{T}$ is a normal C-intelim tableau proof of A from Γ, then $\mathcal{T}$ is non-contaminated or improper (i.e., a closed tableau for Γ).*

One can consider a restricted version of the C-intelim rules that automatically generates normal tableaux (except at most for the presence of idle occurrences of formulae), by requiring that, in the attempt to prove A from Γ:

- an inference rule can be applied in a branch only if its conclusion does not already occur in the branch;
- RB can be applied in a branch only (i) if neither of the virtual assumptions introduced by it already occurs in the branch, and (ii) the RB formula is a subformula either of one of the premises in Γ or of the conclusion A;
- a closed branch cannot be further expanded.

A tableau $\mathcal{T}$ constructed in accordance with these restrictions can be easily turned into a normal one by simply removing all idle occurrences of formulae (if any) and then, if required, into a normal C-intelim proof in the upward tree format of Section 1.

[12] As in Definitions 1.1 and 1.2 of Part I, an application of RB in a C-intelim tableau proof of A from Γ (refutation of Γ) is *analytic* when its RB formula is a subformula of some formula in $\Gamma \cup \{A\}$ (in Γ) and *atomic* when its RB formula is atomic.

5 The complexity of C-intelim proofs

Let QNC-intelim be the restriction of C-intelim to quasi-normal proofs.

Theorem 5.1. *QNC-intelim can p-simulate Frege systems.*

Hint. As the reader can verify, the negation of any instance A of any axiom scheme of a typical complete axiomatic systems for classical propositional logic admits of a 0-depth C-intelim refutation. So, by means of an application of RB one can obtain a proof of A as follows:

$$\dfrac{[A]^1 \qquad \dfrac{\begin{matrix}[\neg A]^2 \\ \mathcal{T} \\ \curlywedge\end{matrix}}{A}}{A}\,{}_{1,2}$$

Moreover, Modus Ponens is a rule of C-intelim. □

Consider now normal C-intelim tableaux. Since the introduction rules can be easily simulated by means of eliminations and applications of RB, it is not difficult to show that the subsystem of C-intelim tableaux consisting only of RB and the elimination rules, which amounts to the KE system of D'Agostino and Mondadori [1994], can linearly simulate C-intelim tableaux and vice versa. Since KE can p-simulate the cut-free Gentzen sequent calculus, but not vice versa [D'Agostino and Mondadori, 1994], it immediately follows that:

Theorem 5.2. *C-intelim tableaux can p-simulate cut-free Gentzen systems, but cut-free Gentzen systems cannot p-simulate C-intelim tableaux.*

Let us write $\Gamma \vdash_k^{\mathrm{IET}} A$ to mean that there is a normal C-intelim tableau proof of A from Γ of depth $\leq k$, and $\Gamma \vdash_k^{\mathrm{IET}} \curlywedge$ to mean that there is a normal refutation of Γ of depth $\leq k$. In [D'Agostino et al., 2013, D'Agostino, 2015] it is shown that each $\vdash_k^{\mathrm{IET}}$ is a tractable approximation to classical propositional logic that converges to it for $k \to \infty$.

Theorem 5.3. *For each $k \in \mathbb{N}$, whether or not $\Gamma \vdash_k^{\mathrm{IET}} A$ can be decided in time $O(n^{k+2})$, where n is the total number of occurrences of symbols in $\Gamma \cup \{A\}$.*

Let *NC-intelim* be the restriction of C-intelim to normal proofs and *NC-intelim*$_k$ be the restriction of NC-intelim in which only proofs of depth $\leq k$ are allowed. Given the correspondence between NC-intelim proofs and C-intelim tableaux discussed in Section 4, it follows from the results above that proof search in NC-intelim$_k$ is feasible.

6 Conclusions

We have carried out a detailed proof-theoretical study of a system of natural deduction for classical propositional logic, the C-intelim system, that considerably departs from the standard Gentzen-style approach in that classical logic is not characterized by a system of intelim rules that extend the rules that represent the intuitionistic meaning of the logical operators, but directly by means of rules that are faithful to their classical interpretation. As a result, the typical symmetries of classical logic are not lost, no intelim rule involves the discharge of assumptions and the only discharge rule is the one that expresses the classical Principle of Bivalence (often called "Classical Dilemma" Tennant [1990]). We have shown sets of transformations that, independently of the order in which they are applied, yield respectively *quasi-normal proofs* — i.e. non-analytic proofs that, yet, contain no trivial detours — and *normal proofs* that are fully analytic and enjoy the (weak) subformula property. We have also introduced *proper normal proofs*, that avoid uses of the ex-falso quodlibet principle that are inferentially meaningless, and enjoy the variable sharing property. We have also shown that our notion of normal proof paves the way for the proof of a *non-contamination* theorem that does not hold if the standard notion of normal proof in classical Gentzen-style natural deduction is adopted. This result shows that a weak relevance property that holds for classical logic, i.e., the existence of non-contaminated proofs except for trivial cases in which the proof is more aptly described as a refutation of the assumptions, is automatically enforced by the restriction to normal proofs. Next, we have presented a different format for C-intelim proofs that is more suitable to provide a more concise representation of proofs and to implement proof-search algorithms, highlighting the connection between C-intelim and certain variants of the method of semantic tableaux. Finally we have stated some complexity facts about C-intelim and in particular that the notion k-depth C-intelim deducibility provides a hierarchy of tractable approximation to classical propositional logic. This approach to tractable depth-bounded reasoning will be extended to first-order logic in a subsequent paper, by incorporating some ideas from Hintikka [1972].

We maintain that our results are useful in a variety of application areas. In particular, a key application area of increasing prominence is the use of argumentation theory to formalise individual agent, and distributed, non-monotonic reasoning. Typically, arguments are classical logic proofs possibly augmented by defeasible inference rules [Modgil and Prakken, 2013]. Evaluation of the interacting (attacking and counter-attacking) arguments determines those that are justified, and the conclusions of these justified arguments identify the inferences from the theory that supplies the assumptions for constructing the arguments. Two key reasons for the increasing prominence of argumentation theory are that: 1) its characterisation of non-monotonic reasoning, in terms of argument and counter-argument, is intuitively understandable to human users familiar with everyday principles of debate and discussion; 2) it paves the way for practical applications of individual and distributed non-monotonic reasoning accommodating both computational and human agents.

The perspicuity of natural deduction makes it an obvious proof-theoretic choice for constructing the deductive parts of arguments, especially in light of argumentation theory's aim to make computational reasoning transparent for human inspection and interaction. We suggest that our more natural representation of classical proofs (as compared with standard Gentzen-style classical natural deduction) further supports this rationale for natural deduction representations of the deductive parts of arguments.

Moreover, as explored in D'Agostino and Modgil [2016, 2017], our formulation of NC-intelim$_k$ proofs has important implications for practical applications of argumentation, and more generally for applications of logic to modelling the inferential behaviour of non-ideal agents. Firstly, the computational tractability of constructing k-depth proofs can be exploited for use by real-world agents with limited inferential capabilities; each increase in depth naturally equates with the inferential resources that agents deploy in constructing proofs. Indeed, D'Agostino and Modgil [2017] show that key rationality postulates for argumentation [Caminada and Amgoud, 2007, Caminada et al., 2012], previously shown only under the assumption that agents have unbounded resources, are in fact satisfied for agents reasoning to any given depth k. Secondly, evaluation of arguments yields counterintuitive results, and violates rationality postulates, if arguments include assumptions that are redundantly used in deriving the conclusion, i.e., arguments that are contaminated. For example, a redundant assumption may inappropriately be accounted for in determining the weight/strength of an argument and hence its evaluation. Now, the standard approach to solving this problem is to ensure that arguments are not contaminated by verifying that an argument is valid only if it satisfies the stronger property that no proper subset of an argument's assumptions suffice to entail the conclusion. Clearly this is computationally impractical. Hence, there are good practical and theoretical reasons for ensuring that only non-contaminated arguments are delivered by a natural deduction proof theory. Again, D'Agostino and Modgil [2017] show that all rationality postulates hold when arguments are formalised as NC-intelim$_k$ proofs, and without requiring that one check that the arguments' assumptions are subset minimal.

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