Exercises on derivations
in modal propositional logic (S5)

Sandro Zucchi

2022-23

Some examples

Before we introduce the exercises, let's see some examples of derivations in S5(NAT). We begin by showing that

$$\vdash_{S5(NAT)} \Box p \supset p$$

The derivation is immediate, given the rule of $\Box$ elimination:

1.	~~Prova~~: $\Box p \supset p$	$\supset$I
2.	$\Box p$	Ass
3.	p	$\Box$E, 2

Now let's derive some examples of necessitation of formulae that are valid in proposition logic:

$$\vdash_{S5(NAT)} \Box(p \supset p)$$
$$\vdash_{S5(NAT)} \Box(p \vee \sim p)$$

1.	~~Prova~~: $\Box(p \supset p)$	$\Box$I
2.	~~Prova~~: $p \supset p$	$\supset$I
3.	p	Ass
4.	p	R, 3

1.	~~Prova~~: $\Box(p\vee\sim p)$	$\Box$I
2.	~~Prova~~: $p\vee\sim p$	$\sim$E
3.	$\sim (p\vee\sim p)$	Ass
4.	~~Prova~~: $\sim p$	$\sim$I
5.	p	Ass
6.	$p\vee\sim p$	$\vee$I, 5
7.	$\sim (p\vee\sim p)$	R,3
8.	$p\vee\sim p$	$\vee$I, 4

The last two derivations show how, given a formula φ which is derivable in LP(NAT) without any auxiliary premise, we can prove $\Box\varphi$ in S5(NAT). Indeed, since the rules of LP(NAT) are also rules of S5(NAT), we may insert the derivation of φ by the rules of LP(NAT) in the box of the derivation of $\Box\varphi$ by $\Box$I in S5(NAT). Since the set of formulae derivable in LP(NAT) without any premise coincides with the set valid of formulae in LP(NAT) and the set of formulae derivable in S5(NAT) without any premise coincides with the set valid of formulae in S5(NAT), an immediate consequence is that if φ is a valid formula in LP, then $\Box\varphi$ is valid in LS5.

Now let's see an example of a derivation in S5(NAT) by $\Diamond$E:

$$\vdash_{S5(NAT)} \Diamond p \supset \Diamond(p\vee \sim p)$$

1.	~~Prova~~: $\Diamond p \supset \Diamond(p \vee \sim p)$	$\supset$I
2.	│ $\Diamond p$	Ass
3.	│ ~~Prova~~: $\Diamond(p \vee \sim p))$	$\Diamond$E, 2
4.	│ │ p	Ass
5.	│ │ $p \vee \sim p$	$\vee$I, 4

Now let's prove that if p is necessary, then necessarily p is necessary:

$\vdash_{S5(NAT)} \Box p \supset \Box\Box p$

1.	~~Prova~~: $\Box p \supset \Box\Box p$	$\supset$I
2.	│ $\Box p$	Ass
3.	│ ~~Prova~~: $\Box\Box p$	$\Box$I
4.	│ │ $\Box p$	R, 2

Notice that the application of the reiteration rule R at line 4 is permitted, since line 4 is available and formulae of the form $\Box\varphi$ can be imported by R in a proof by $\Box$I.

Now let's prove that if p is possible, then necessarily p is possible:

$\vdash_{S5(NAT)} \Diamond p \supset \Box\Diamond p$

1.	~~Prova~~: $\Diamond p \supset \Box\Diamond p$	$\supset$I
2.	│ $\Diamond p$	Ass
3.	│ ~~Prova~~: $\Box\Diamond p$	$\Box$I
4.	│ │ $\Diamond p$	R, 2

Notice that the application of the reiteration rule R at line 4 is permitted, since line 4 is available and formulae of the form $\Diamond\varphi$ can be imported by R in a proof by $\Box$I.

First exercise

Prove the following claims:

(1) a. $\vdash_{S5(NAT)} \Box(p \supset q) \supset (\Box p \supset \Box q)$
b. $\vdash_{S5(NAT)} \Box\Box p \supset \Box p$

c. $\vdash_{S5(NAT)} \Box p \supset \Diamond p$
d. $\vdash_{S5(NAT)} \Box(p \supset \Diamond p)$
e. $\Box(p \supset q) \vdash_{S5(NAT)} \Diamond p \supset \Diamond q$

f. $\vdash_{S5(NAT)} \Box\Box\Box p \equiv \Box p$
g. $\vdash_{S5(NAT)} \Diamond\Diamond p \equiv \Diamond p$
h. $\vdash_{S5(NAT)} \Diamond\Diamond\Diamond p \equiv \Diamond p$
i. $\vdash_{S5(NAT)} \Box\Diamond\Diamond p \supset \Box\Diamond p$
j. $\vdash_{S5(NAT)} \Box\Diamond p \supset \Box\Diamond\Diamond p$

k. $\vdash_{S5(NAT)} \Diamond\Box p \equiv \Box p$
l. $\vdash_{S5(NAT)} \Diamond\Box p \supset \Box\Box p$
m. $\vdash_{S5(NAT)} \Diamond\Box p \supset p$
n. $\Diamond p, \Box(p \supset \Box p) \vdash_{S5(NAT)} \Box p$

Second exercise

1. The following proof contains a mistake. Where is the mistake?

1.	$\sim \Diamond p$	P
2.	$q \supset p$	P
3.	~~Prova~~: $\sim \Diamond q$	$\sim$I
4.	$\Diamond q$	Ass
5.	~~Prova~~: $\Diamond p$	$\Diamond$E, 4
6.	q	Ass
7.	$q \supset p$	R, 2
8.	p	$\supset$E, 6, 7
9.	$\sim \Diamond p$	R, 1