A quick tutorial on de Finetti’s coherence

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Coherence under uncertainty

Building on core decision-theoretic ideas (then not explicitly systematised!) de Finetti sets up a choice problem which reduces the probability of an event to the choice of a fair price for the corresponding gamble.

The key idea consists in defining rational choice as avoiding blatantly irrational behaviour, which in turn is defined in terms of making sure that one does not put oneself in a betting situation in which, no matter how the relevant uncertainty is resolved, one can be led to a monetary loss.

The betting scheme

Three steps

1. Probability as price: i’s degree of belief in θ is identified with their willingness to make certain transactions on θ

2. Rationality as coherence: irrationality is identified with i’s willingness to face sure loss, and rationality avoiding that

3. The theorem proof that it is necessary and sufficient for i’s rationality that the opinions revealed by their choice behaviour comply with the laws of probability
Example

Suppose $\theta \equiv \neg\phi$. And suppose that Bookmaker assigns distinct prices (in cents) for the corresponding bets

$$X_\theta = \begin{cases} 100 & \text{if } v(\theta) = 1, \\ 0 & \text{otherwise} \end{cases}$$

$$X_{\neg\phi} = \begin{cases} 100 & \text{if } v(\phi) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Suppose for instance that $p_\theta = 20$ e $p_{\neg\phi} = 70$. By buying both gambles we can lead Bookmaker into sure loss.

- Buying $X_\theta$ at price $p_\theta$ we lose 20 cents to receive 100 if $v(\theta) = 1$ (net balance of +80c.) and lose 20c. to gain nothing if $v(\theta) = 0$ (net balance of -20c.)

- Buying $X_{\neg\phi}$ at price $p_{\neg\phi}$ we lose -70c. to receive 100 if $v(\neg\phi) = 1$ (net balance of +30c.) and lose 70c. to gain nothing if $v(\neg\phi) = 0$ (net balance of -70c.)

Table 1 shows that if Bookmaker allows us to buy both bets they will end with a negative balance:

<table>
<thead>
<tr>
<th></th>
<th>$v(\theta) = 1$</th>
<th>$v(\phi) = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_\theta$</td>
<td>+80</td>
<td>-20</td>
</tr>
<tr>
<td>$X_{\neg\phi}$</td>
<td>-70</td>
<td>+30</td>
</tr>
<tr>
<td>TOT</td>
<td>10</td>
<td>10</td>
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</tbody>
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The model setup

De Finetti’s construction starts with the definition of a zero sum game between Bookmaker (Bo) and Gambler (Gam) about which the following hypothesis is made

**Agent Idealisation**

**A1** Bo and Gam are **logically omniscient**

**A2** Bo and Gam are **economically rational** i.e. they are completely described by their consistent preference relation $\succeq_B$ e $\succeq_G$

- The goal is to make sure that Bo assigns coherently to a set of event of interests $\theta_1, \ldots, \theta_n$ a number $p_i \in [0,1]$ with $i = 1, \ldots, n$.

- for this we need further constraints: the rules of the zero sum game
The rules of the game

1. Bo starts by choosing, for each $\theta_i$, a number $p_i \in [0, 1]$ interpreted as the betting quotient for $\theta$

   - for simplicity let us consider a single $\theta$ and the $p$ Bo assigns to it

2. Gam then chooses an $S \in \mathbb{R}$ and one between the following gambles

   $F_p(\theta) = \begin{cases} S(1 - p) & \text{if } v(\theta) = 1 \\ -S(p) & \text{otherwise} \end{cases}$

   and

   $A_p(\theta) = \begin{cases} -S(1 - p) & \text{if } v(\theta) = 1 \\ S(p) & \text{otherwise} \end{cases}$

   - write $F_p(\theta) \succcurlyeq A_p(\theta)$ to say that Gam prefers (hence chooses) $F_p(\theta)$

It is natural to interpret $F_p(\theta)$ as Gam’s choice to bet $S$ monetary units (that is to pay $Sp$ to Bo) on $\theta$ obtaining, and hence to interpret $p$ as a betting quotient

Summing up

1. for each $\theta_i$, Bo chooses $p_i \in [0, 1]$.

2. Gam chooses $S$ and one between $\{F_p(\theta), A_p(\theta)\}$, where

\[
\begin{array}{c|cc}
F_p(\theta) & v(\theta) = 1 & v(\theta) = 0 \\
\hline
S(1 - p) & -Sp \\
\end{array}
\]

\[
\begin{array}{c|cc}
A_p(\theta) & v(\theta) = 1 & v(\theta) = 0 \\
\hline
-S(1 - p) & Sp \\
\end{array}
\]

Immediate consequences

- Suppose $p = 0$. By hyp A1.A2, Gam will certainly choose $F_p(\theta)$

- Suppose $p = 1$. Again A1.A2 force Gam to choose $S_p(\theta)$

- Suppose now that $0 \leq p' < p \leq 1$ and $F_p(\theta) \succcurlyeq A_p(\theta)$. Then A1-A2 imply that $F_{p'}(\theta) \succcurlyeq A_{p'}(\theta)$. Similarly for $0 \leq p > p'' \leq 1$

Probability as price

Denote by $p_\theta$ the sup of the set $\{p \mid F_p(\theta) \succcurlyeq A_p(\theta)\}$.

Reasoning as above we can see that A1.A2 force

- $F_p(\theta) \succcurlyeq A_p(\theta)$ if $p < p_\theta$

- $A_p(\theta) \succcurlyeq F_p(\theta)$ if $p_\theta < p$.

It now follows that the price $p_\theta$ represents Gam’s degree of belief in $\theta$
Rationality as coherence

We now want to use this model to define Bo’s coherence and to do so we need a final set of assumptions

Abstraction on the problem

1. Completeness Once Bo published his book (i.e. the $p_i$’s) he cannot refuse to sell any combination of bets to Gam

2. Swapping By choosing a negative $S$ Gam can unilaterally impose a payoff swap to Bo

3. Rigidity $S$ must be small

The modelling assumptions guarantee that the choice problem is adequate a "device to force the individual to make conscious choices, releasing him from inertia, preserving him from whim"
Coherence and fair prices
Idalisation on agents and the abstraction on the problem make the betting framework essentially equivalent to a fair cake cutting problem where

1. Bo cuts the cake
2. Gam chooses the slice

Key idea
Avoiding sure loss (under those conditions!) means assigning fair prices to events, i.e. a book with null expectation

Subjective Probability Norm Justified

Definition 0.1. Let $B = \{\theta_1, \ldots, \theta_k\} \subseteq SL$. The book $\beta_i : \theta_i \mapsto p_i \in [0,1]$ $i = 1, \ldots, k$ is coherent iff there exist no $S_1, \ldots, S_k \in \mathbb{R}$ such that for every $v \in V$

$$\sum_{i=1}^{n} S_i(p_i - v(\theta_i)) < 0.$$ 

Theorem 0.2 (De Finetti 1931). An assessment $\theta_1 \mapsto p_1, \ldots, \theta_k \mapsto p_k$ is coherent iff it can be extended to a probability distribution, that is, there exists a function $P : SL \rightarrow [0,1]$ such that, for $i = 1, \ldots, k$, $P(\theta_i) = p_i$.  

5
On the Modelling Assumptions
The result to the effect that coherent degrees of belief must be ‘fair’ betting odds, i.e. be consistent with probability is derived under strong modelling assumptions

Agent
1. logically infallible
2. ‘economic rationality’

Problem
1. completeness
2. swapping
3. rigidity

An exciting new research area
Can we justify more general norms of rational belief by relaxing some of those assumptions?

Much confusion about this topic
What is the methodological (modelling) role of de Finetti’s betting problem?

• metaphor?
• elicitation device?
• definition?
• meaning?

Decision modelling
Effective way to map our intuitions about (ir)rational behaviour to a mathematically well-defined choice problem, which as a bonus, justifies the inevitability of measuring uncertainty with probability

For the details

  – The real thing
– General, cristal-clear proof (in logic)


– A glimpse at the key new challenges

Take-home points

• De Finetti’s Dutch Book Argument provides a justification for relating probability with the rational measurement of uncertainty

• The argument is rooted in choice-theoretic consistency

• Strongly influenced by Paretian ordinalism

• Heavy duty for strong modelling assumptions

• de Finetti’s argument is non-constructive: it does not provide a way of choosing actual degrees of belief